FUNCTIONS AS SET BEHAVIOR

Abstract. The term *function* suggests an action or process or behavior of something applied to something. Within the framework of extended set theory, XST, the concept of a *function* will be defined in terms of a liberal definition of *morphism*. Which in turn will be equated with the behavior of how specific sets interact with each other. It will be shown that all Classical set theory, CST, graph based function behavior can be expressed in terms of XST function non-graph based behavior; that the behavior of functions applied to themselves is supported; and that the concepts of Category theory can be subsumed under XST. A notable consequence of this approach is that the use of functions need no longer be constrained by properties of a Cartesian product.

1. Introduction

This paper does not purport to present anything mathematically new. It does, however, attempt to provide a more comprehensive environment for expressing and relating what is already mathematically familiar regarding functions.

The only deviation from traditional definitions is in not defining functions as specific sets, but rather as the *behavior* of specific sets. The extended definition of function in terms of *unique set behavior* instead of being defined in terms of *unique set membership* avoids many problems emanating from graph based definitions. Specifically, the classical concern with the *self-application* of functions is avoided by focusing on the behavior of sets instead of on their anatomy. The elimination of a graph based dependence also allows for a richer definition of *morphism* allowing a strictly set-theoretic model of Categories.

2. Morphisms

For both conceptual convenience and mathematical compatibility, the term **morphism** will be defined as an abstract modeling symbol that can be algebraically manipulated and combined with other like symbols to reflect the behavior of one set as influenced by another. Not having any mathematical substance, morphisms do not exist in any formal set theory and thus can not be contained in sets. However, since the notation for a morphism can be defined in terms of legitimate sets, $\mathbf{f}$ and σ, sets of morphisms can be simulated by sets of the form $\{\mathbf{f}^{\sigma}\}$.

Definition 2.1. Morphism: Two sets $\mathbf{f}$ and σ define a morphism $\mathbf{f}_{(\sigma)}$ iff

$$(\exists x)\big(\ \mathbf{f}_{(\sigma)}(x) \neq \varnothing\big)\ \&\ (\forall \mathbf{g})(\mathbf{g} \Subset \mathbf{f})(\exists x)\big(\ \mathbf{g}_{(\sigma)}(x) \neq \varnothing\ \big).$$

Definition 2.2. Morphism Equality:

$$\mathbf{f}_{(\sigma)} = \mathbf{g}_{(\omega)} \iff (\forall x)(\ \mathbf{f}_{(\sigma)}(x) = \mathbf{g}_{(\omega)}(x)\ ).$$

Since all uses of morphism are only a prediction, the actual set behavior can not be realized until a morphism is given a set-theoretic interpretation.

Given that $\mathbf{f}\big[x\big]_{\sigma}$ is a well defined set-theoretic description, the following definition of *application* transforms a morphism into a set-theoretic reality.

Definition 2.3. Application: $\mathbf{f}_{(\sigma)}(x) \ = \ \mathbf{f}\big[x\big]_{\sigma}$.

Consequence 2.1. *Application Properties:*

$$\begin{aligned}
&(a)\ \big(\mathbf{f} \cup \mathbf{g}\big)_{(\sigma)}(x) \ &=&\ \ \mathbf{f}_{(\sigma)}(x) \ \cup\ \mathbf{g}_{(\sigma)}(x),\\
&(b)\ \big(\mathbf{f} \cap \mathbf{g}\big)_{(\sigma)}(x) \ &\subseteq&\ \ \mathbf{f}_{(\sigma)}(x) \ \cap\ \mathbf{g}_{(\sigma)}(x),\\
&(c)\ \mathbf{f}_{(\sigma)}(x) \ \sim\ \mathbf{g}_{(\sigma)}(x) \ &\subseteq&\ \ \big(\mathbf{f} \sim \mathbf{g}\big)_{(\sigma)}(x).
\end{aligned}$$

Application dictates a set behavior that produces a result set, when the behavior is applied to a set. It is important to notice that the application of a morphism produces a set while the morphism itself defines a behavior of a set, and does not define a specific set.

Note: '$\mathbf{f}_{(\sigma)}(x) \in_s \mathbf{Q}$' makes mathematical sense, while '$\mathbf{f}_{(\sigma)} \in_s \mathbf{Q}$' does not. The expression '$\mathbf{f}_{(\sigma)}(x)$' defines a set-membership condition, while the expression '$\mathbf{f}_{(\sigma)}$' defines a set-behavior.

Definition 2.4. Nested Application: $\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)}) = \Big(\mathbf{f}_{(\sigma)}(\mathbf{g})\Big)_{(\omega)} = \Big(\mathbf{f}\left[\mathbf{g}\right]_{\sigma}\Big)_{(\omega)}$.

Though this is not the only way to subvert the application notation into accommodating functions applied to functions, it is simple, somewhat natural, and does produce meaningful behaviors. Notice that $\mathbf{g}_{(\omega)}$ may or may not make sense, but whether or not it does is not relevant to the definition. Note also a application applied to a morphism produces another morphism, not a result set!

Though application is well defined, sequences of applications are not. Consider the simple expression $\mathbf{f}_{(\sigma)}\mathbf{g}_{(\omega)}(x)$. Without proper bracketing or an explicitly defined bracketing convention, its meaning is ambiguous. There are two legitimate interpretations: $\mathbf{f}_{(\sigma)}\Big(\mathbf{g}_{(\omega)}(x)\Big)$ and $\Big(\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)})\Big)(x)$. The differences are explicit in the following example.

Example 2.1. *Interpretations of* $\mathbf{f}_{(\sigma)}\mathbf{g}_{(\omega)}(x)$.

$$(a)\quad \mathbf{f}_{(\sigma)}\Big(\mathbf{g}_{(\omega)}(x)\Big) \;=\; \mathbf{f}\left[\mathbf{g}\left[x\right]_{\omega}\right]_{\sigma},$$

$$(b)\quad \Big(\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)})\Big)(x) \;=\; \Big(\mathbf{f}_{(\sigma)}(\mathbf{g})\Big)_{(\omega)}(x) \;=\; \Big(\mathbf{f}\left[\mathbf{g}\right]_{\sigma}\Big)\left[x\right]_{\omega}.$$

Example 2.2. *Interpretations of* $\mathbf{f}_{(\sigma)}\mathbf{g}_{(\omega)}\mathbf{h}_{(\tau)}(x)$.

$$(a)\quad \mathbf{f}_{(\sigma)}\Big(\mathbf{g}_{(\omega)}\Big(\mathbf{h}_{(\tau)}(x)\Big)\Big) \;=\; \mathbf{f}\Big[\mathbf{g}\left[\mathbf{h}\left[x\right]_{\tau}\right]_{\omega}\Big]_{\sigma},$$

$$(b)\quad \mathbf{f}_{(\sigma)}\Big(\big(\mathbf{g}_{(\omega)}(\mathbf{h}_{(\tau)})\big)(x)\Big) \;=\; \mathbf{f}_{(\sigma)}\Big(\big(\mathbf{g}_{(\omega)}(\mathbf{h})\big)_{(\tau)}(x)\Big) \;=\; \mathbf{f}\Big[\Big(\mathbf{g}\left[\mathbf{h}\right]_{\omega}\Big)\left[x\right]_{\tau}\Big]_{\sigma},$$

$$(c)\quad \mathbf{f}_{(\sigma)}\Big(\mathbf{g}_{(\omega)}(\mathbf{h}_{(\tau)})\Big)(x) \;=\; \mathbf{f}_{(\sigma)}\Big(\big(\mathbf{g}_{(\omega)}(\mathbf{h})\big)_{(\tau)}\Big)(x) \;=\; \mathbf{f}\Big[\mathbf{g}\left[\mathbf{h}\right]_{\omega}\Big]_{\sigma}\left[x\right]_{\tau},$$

$$(d)\quad \Big(\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)})(\mathbf{h}_{(\tau)})\Big)(x) \;=\; \Big(\big(\mathbf{f}_{(\sigma)}(\mathbf{g})\big)_{(\omega)}(\mathbf{h})\Big)_{(\tau)}(x) \;=\; \Big(\mathbf{f}\left[\mathbf{g}\right]_{\sigma}\left[\mathbf{h}\right]_{\omega}\Big)\left[x\right]_{\tau},$$

$$(e)\quad \Big(\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)})\Big)\Big(\mathbf{h}_{(\tau)}(x)\Big) \;=\; \Big(\mathbf{f}_{(\sigma)}(\mathbf{g})\Big)_{(\omega)}\Big(\mathbf{h}_{(\tau)}(x)\Big) \;=\; \mathbf{f}\left[\mathbf{g}\right]_{\sigma}\left[\mathbf{h}\left[x\right]_{\tau}\right]_{\omega}.$$

Note: Interpretations for a sequences greater than three gets large rather quickly with 14 for four and 42 for five.

It may not be immediately apparent that even the simplest case has more that one valid interpretation. Therefore it must be shown that there is a case where both interpretations are non-empty and not equal to each other, (see appendix A).

3. XST & CST Equivalences

Since the definition of morphism is independent of any specific choice of supporting set theory, one has to be chosen for any set behavior to be realized. The choice of set theory chosen for the rest of this paper follows the axioms of an extended set theory[1].

Given that there is an association (application) between a linguistic notion of morphism and set behavior, it is not a foregone conclusion that all the familiar set operations behave predictably when expressed in terms of morphisms and morphism expressions.

Since XST definitions will be used by the supporting set theory, their similarities and differences with CST definitions need to be established.

Definition 3.1. Re-Scope by Scope:
$\mathbf{a}^{/\sigma/} \;=\; \{\mathbf{x}^w \;\; (\exists s)\big(\; \mathbf{x} \in_s \mathbf{a} \;\;\&\;\; s \in_w \sigma\;\big)\}.$

Definition 3.2. Re-Scope by Element:
$\mathbf{a}^{\backslash\sigma\backslash} \;=\; \{\mathbf{x}^w \;\; (\exists s)\big(\; \mathbf{x} \in_s \mathbf{a} \;\;\&\;\; w \in_s \sigma\;\big)\}.$

These two definitions are duals. They both replace scope values of a given set. One with scope values from another set. The other with element values from another set.

For example:

$$\{a^A, b^B, c^C\}^{/\{A^X,B^Y,C^Z\}/} = \{a^X, b^Y, c^Z\},$$
$$\{a^A, b^B, c^C\}^{\backslash\{Q^A,R^B,S^C\}\backslash} = \{a^Q, b^R, c^S\}.$$

Definition 3.3. Ordered Pair: $< x, y > \;=\; \{\; x^1, y^2 \;\}.$

Definition 3.4. Domain Extraction:

$$\mathbf{Đ}_\sigma(\mathbf{Q}) = \Big\{\; \mathbf{x}^s\colon (\exists \mathbf{z}, w)\big(\; \mathbf{z} \in_w \mathbf{Q} \;\;\&\; \mathbf{x} = \mathbf{z}^{/\sigma/} \neq \varnothing \;\;\&\; s = w^{/\sigma/}\;\big)\Big\}.$$

For example:

$$\mathbf{Đ}_{\{A^1,C^2\}}(\{\{a^A, b^B, c^C\}\}) = \{\{a^1, c^2\}\},$$
$$\mathbf{Đ}_{<3,1>}(\{\{a^1, b^2, c^3\}^{\{A^1,B^2,C^3\}}\}) = \{< c, a >^{<C,A>}\},$$
$$\mathbf{Đ}_{\{3^1,1^2,y^9,v^5,v^7,Q^A\}}(\{\{a^1, b^2, c^3\}^{\{x^y,w^v,z^Q\}}\}) = \{< c, a >^{\{x^9,w^5,w^7,z^A\}}\}.$$

The usual CST properties associated with the Domain operation are preserved in XST.

Consequence 3.1. *Preserved Domain Properties:*

(a) $\mathbf{Đ}_\sigma(\mathbf{Q} \cup \mathbf{R}) \;=\; \mathbf{Đ}_\sigma(\mathbf{Q}) \cup \mathbf{Đ}_\sigma(\mathbf{R}),$
(b) $\mathbf{Đ}_\sigma(\mathbf{Q} \cap \mathbf{R}) \;\subseteq\; \mathbf{Đ}_\sigma(\mathbf{Q}) \cap \mathbf{Đ}_\sigma(\mathbf{R}),$
(c) $\mathbf{Đ}_\sigma(\mathbf{Q}) \sim \mathbf{Đ}_\sigma(\mathbf{R}) \;\subseteq\; \mathbf{Đ}_\sigma(\mathbf{Q} \sim \mathbf{R}),$
(d) $\mathbf{Q} \subseteq \mathbf{R} \;\longrightarrow\; \mathbf{Đ}_\sigma(\mathbf{Q}) \subseteq \mathbf{Đ}_\sigma(\mathbf{R}),$
(e) $\mathbf{Đ}_\varnothing(\mathbf{Q}) = \varnothing.$

The following scope definitions will be needed for subsequent definitions

Definition 3.5. Scope Set: $\mathcal{S}(a) = \{s^s : (\exists x)\, x \in_s a\}.$

Definition 3.6. Element Scope Set:

$$\mathcal{S}c(\mathbf{A}) \;=\; \{\; y^y \;\; (\exists z, s)(\; z \in_s \mathbf{A} \;\&\; y \in_y \mathcal{S}(z))\}.$$

EXAMPLE: $\mathcal{S}c(\{\{a^A, b^B, c^C\}, \{x^X, y^Y\}\}) = \{A^A, B^B, C^C, X^X, Y^Y\}.$

Definition 3.7. Restriction:

$$\mathbf{Q}\,|_{\sigma}\,\mathbf{A} = \Big\{\; \mathbf{z}^w\colon \big(\; \mathbf{z} \in_w \mathbf{Q}\big) \;\&\;\; \big(\exists\; a, s\big)\big(a \in_s \mathbf{A} \;\&\; a^{\backslash\sigma\backslash} \subseteq \mathbf{z} \;\;\&\; s^{\backslash\sigma\backslash} \subseteq w\big)\Big\}.$$

The usual CST properties associated with the Restriction operation are preserved in XST. For commensurate behavior with established CST notation, the following definition is introduced.

Definition 3.8. $\mathbf{Q}\,|\,\mathbf{A} \;\;=\;\; \mathbf{Q}\,|_{\sigma}\,\mathbf{A}$ *with* $\sigma = \mathcal{S}c(\mathbf{A}).$

Consequence 3.2. *Preserved Restriction Properties:*

(a) $\mathbf{Q}\,|_{\sigma}(\mathbf{A} \cup \mathbf{B}) \;=\; \mathbf{Q}\,|_{\sigma}\,\mathbf{A} \;\cup\; \mathbf{Q}\,|_{\sigma}\,\mathbf{B},$
(b) $\mathbf{Q}\,|_{\sigma}(\mathbf{A} \cap \mathbf{B}) \;\subseteq\; \mathbf{Q}\,|_{\sigma}\,\mathbf{A} \;\cap\; \mathbf{Q}\,|_{\sigma}\,\mathbf{B},$
(c) $\mathbf{Q}\,|_{\sigma}\,\mathbf{A} \sim \mathbf{Q}\,|_{\sigma}\,\mathbf{B} \;\subseteq\; \mathbf{Q}\,|_{\sigma}(\mathbf{A} \sim \mathbf{B}),$
(d) $\mathbf{A} \subseteq \mathbf{B} \;\longrightarrow\; \mathbf{Q}\,|_{\sigma}\,\mathbf{A} \subseteq \mathbf{Q}\,|_{\sigma}\,\mathbf{B},$

(e) $\mathbf{Q}\,|_{\sigma}(\text{Đ}_\sigma(\mathbf{Q}) \cap \mathbf{A}) \;=\; \mathbf{Q}\,|_{\sigma}\mathbf{A}$,
(f) $\mathbf{Q}\,|_{\sigma}\mathbf{A} \;\subseteq\; \mathbf{Q}$,
(g) $\mathbf{Q}\,|_{\sigma}\varnothing = \varnothing \;\;\&\;\; \mathbf{Q}\,|_{\varnothing}\mathbf{A} = \varnothing$,
(h) $\text{Đ}_\sigma(\mathbf{Q}) \cap \mathbf{A} = \varnothing \;\longrightarrow\; \mathbf{Q}\,|_{\sigma}\mathbf{A} = \varnothing$.
(i) $(\mathbf{Q} \cup \mathbf{R})\,|_{\sigma}\mathbf{A} \;=\; \mathbf{Q}\,|_{\sigma}\mathbf{A} \cup \mathbf{R}\,|_{\sigma}\mathbf{A}$,
(j) $(\mathbf{Q} \cap \mathbf{R})\,|_{\sigma}\mathbf{A} \;\subseteq\; \mathbf{Q}\,|_{\sigma}\mathbf{A} \cap \mathbf{R}\,|_{\sigma}\mathbf{A}$,
(k) $\mathbf{Q}\,|_{\sigma}\mathbf{A} \sim \mathbf{R}\,|_{\sigma}\mathbf{A} \;\subseteq\; (\mathbf{Q} \sim \mathbf{R})\,|_{\sigma}\mathbf{A}$,
(l) $\mathbf{Q} \;\dot{\subseteq}\; \mathbf{R} \;\longrightarrow\; \mathbf{Q}\,|_{\sigma}\mathbf{A} \;\subseteq\; \mathbf{R}\,|_{\sigma}\mathbf{A}$,
(m) $\mathbf{Q}\,|\,\mathbf{Q} \;=\; \mathbf{Q}$.

Though Image can be defined directly, its behavior is more intuitive if recognized as a two step process.

Definition 3.9. Image: $\mathbf{Q}\big[\mathbf{A}\big]_{<\sigma_1,\sigma_2>} = \text{Đ}_{\sigma_2}(\mathbf{Q}\,|_{\sigma_1}\mathbf{A})$.

The usual CST properties associated with the Image operation are preserved in XST.

Consequence 3.3. *Preserved Image Properties:*
(a) $\mathbf{Q}\big[\mathbf{A} \cup \mathbf{B}\big]_\sigma \;=\; \mathbf{Q}\big[\mathbf{A}\big]_\sigma \,\cup\, \mathbf{Q}\big[\mathbf{B}\big]_\sigma$,
(b) $\mathbf{Q}\big[\mathbf{A} \cap \mathbf{B}\big]_\sigma \;\subseteq\; \mathbf{Q}\big[\mathbf{A}\big]_\sigma \,\cap\, \mathbf{Q}\big[\mathbf{B}\big]_\sigma$,
(c) $\mathbf{Q}\big[\mathbf{A}\big]_\sigma \sim \mathbf{Q}\big[\mathbf{B}\big]_\sigma \;\subseteq\; \mathbf{Q}\big[\mathbf{A} \sim \mathbf{B}\big]_\sigma$,
(d) $\mathbf{A} \,\dot{\subseteq}\, \mathbf{B} \;\longrightarrow\; \mathbf{Q}\big[\mathbf{A}\big]_\sigma \;\subseteq\; \mathbf{Q}\big[\mathbf{B}\big]_\sigma$,
(e) $\mathbf{Q}\big[\text{Đ}_\sigma(\mathbf{Q}) \cap \mathbf{A}\big]_{<\sigma,\gamma>} \;=\; \mathbf{Q}\big[\mathbf{A}\big]_{<\sigma,\gamma>}$,
(f) $\mathbf{Q}\big[\mathbf{A}\big]_{<\sigma,\gamma>} \;=\; \text{Đ}_\gamma(\mathbf{Q}\,|_{\sigma}\mathbf{A})$.
(g) $\mathbf{Q}\big[\varnothing\big]_\sigma = \varnothing, \quad \varnothing\big[\mathbf{A}\big]_\sigma = \varnothing, \quad \mathbf{Q}\big[\mathbf{A}\big]_\varnothing = \varnothing$,
(h) $\text{Đ}_\sigma(\mathbf{Q}) \cap \mathbf{A} = \varnothing \;\longrightarrow\; \mathbf{Q}\big[\mathbf{A}\big]_{<\sigma,\gamma>} = \varnothing$,
(i) $(\mathbf{Q} \cup \mathbf{R})\big[\mathbf{A}\big]_\sigma \;=\; \mathbf{Q}\big[\mathbf{A}\big]_\sigma \,\cup\, \mathbf{R}\big[\mathbf{A}\big]_\sigma$,
(j) $(\mathbf{Q} \cap \mathbf{R})\big[\mathbf{A}\big]_\sigma \;\subseteq\; \mathbf{Q}\big[\mathbf{A}\big]_\sigma \,\cap\, \mathbf{R}\big[\mathbf{A}\big]_\sigma$,
(k) $\mathbf{Q}\big[\mathbf{A}\big]_\sigma \sim \mathbf{R}\big[\mathbf{A}\big]_\sigma \;\subseteq\; (\mathbf{Q} \sim \mathbf{R})\big[\mathbf{A}\big]_\sigma$
(l) $\mathbf{Q} \;\dot{\subseteq}\; \mathbf{R} \;\longrightarrow\; \mathbf{Q}\big[\mathbf{A}\big]_\sigma \;\subseteq\; \mathbf{R}\big[\mathbf{A}\big]_\sigma$.

4. Applied Morphisms

Morphisms have been defined as a notational abstraction identifying the interaction of two sets, validated by the existence of a non-null application. To support this modeling abstraction requires definitions associating abstract manipulation of morphisms with a concrete manipulation of sets.

Definition 4.1. Scope Functional:
$\mathbf{Sf}(\mathbf{z}) \iff (\forall x, s, y)(x \in_s \mathbf{z} \;\&\; y \in_s \mathbf{z} \to x = y)$.

Definition 4.2. Morphism Restriction: $\mathbf{f}_{(\sigma)}\,|_{\omega}\mathbf{A} \;=\; \big(\mathbf{f}\,|_{\omega}\mathbf{A}\big)_{(\sigma)}$.

Consequence 4.1. $\big(\mathbf{f}_{(\sigma)}\,|_{\omega}\mathbf{A}\big)(x) \;=\; \big(\mathbf{f}\,|_{\omega}\mathbf{A}\big)\big[x\big]_\sigma$.

For the definition of morphism to be at all useful requires an operation of composition.

Definition 4.3. Morphism Composition:

$$\mathbf{h}_{(\tau)} = \mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)} \iff (\forall x)\big(\ \mathbf{h}\big[x\big]_{\tau} = \mathbf{g}\big[\mathbf{f}\big[x\big]_{\sigma}\big]_{\omega} \ \big).$$

Assertion 4.1. *Associativity:*

$$\mathbf{h}_{(\tau)} \circ \mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)} \ = \ \mathbf{h}_{(\tau)} \circ (\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)}) \ = \ (\mathbf{h}_{(\tau)} \circ \mathbf{g}_{(\omega)}) \circ \mathbf{f}_{(\sigma)}.$$

Assertion 4.2. *Substitution:*

$$\mathbf{h}_{(\tau)} = \ \mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)}, \ \mathbf{g}_{(\omega)} = \mathbf{j}_{(\eta)} \ \ \& \ \ \mathbf{f}_{(\sigma)} = \mathbf{k}_{(\nu)} \ \longrightarrow \ \mathbf{h}_{(\tau)} = \ \mathbf{j}_{(\eta)} \circ \mathbf{k}_{(\nu)}.$$

Consequence 4.2. $\big(\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)}\big)(x) \ = \ \mathbf{g}\big[\mathbf{f}\big[x\big]_{\sigma}\big]_{\omega} \ = \ \mathbf{g}_{(\omega)}\Big(\mathbf{f}_{(\sigma)}(x)\Big).$

Consequence 4.3. $(\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)})\,\big|_{\alpha}\mathbf{A} \ = \ \mathbf{g}_{(\omega)} \circ \big(\mathbf{f}\,\big|_{\alpha}\mathbf{A}\big)_{(\sigma)}.$

These definitions make mathematical sense only when the compositions exists, they give no assurance that a specific composition will actually exist.

Assertion 4.3. $(\forall x)\Big(\mathbf{g}_{(\omega)}\big(\mathbf{f}_{(\sigma)}(x) \ \big) \neq \varnothing \ \implies \ \boldsymbol{Đ}_{\sigma_2}(\mathbf{f}) \ \mathrel{\dot\subseteq} \ \boldsymbol{Đ}_{\omega_1}(\mathbf{g})\Big).$

As the name implies, a *submorphism* is much like the concept of subset except that it is defined by set behavior instead of by set membership.

Definition 4.4. Submorphism:

$$\mathbf{g}_{(\sigma)} \sqsubseteq \mathbf{f}_{(\omega)} \iff \boldsymbol{Đ}_{\sigma_1}(\mathbf{g}) \ \mathrel{\dot\subseteq} \ \boldsymbol{Đ}_{\omega_1}(\mathbf{f}) \ \ \& \ \ \boldsymbol{Đ}_{\sigma_2}(\mathbf{g}) \ \mathrel{\dot\subseteq} \ \boldsymbol{Đ}_{\omega_2}(\mathbf{f}) \ \ \&$$
$$(\forall x)\big(\ \mathbf{g}_{(\sigma)}(x) \neq \varnothing \ \longrightarrow \ \mathbf{g}_{(\sigma)}(x) = \mathbf{f}_{(\omega)}(x) \ \big).$$

Consequence 4.4. *Submorphism Properties:*

(*a*) $\mathbf{f}_{(\sigma)} = \mathbf{g}_{(\omega)} \iff \mathbf{g}_{(\sigma)} \sqsubseteq \mathbf{f}_{(\omega)} \ \ \& \ \ \mathbf{f}_{(\omega)} \sqsubseteq \mathbf{g}_{(\sigma)},$

(*b*) $\mathbf{g} \ \mathrel{\dot\subset} \ \mathbf{f} \ \rightarrow \ \mathbf{g}_{(\sigma)} \ \sqsubseteq \ \mathbf{f}_{(\sigma)},$

(*c*) $\mathbf{g}_{(\mu)} \ \sqsubseteq \ \mathbf{f}_{(\sigma)} \ \ \& \ \ \mathbf{f}_{(\sigma)} \ \sqsubseteq \ \mathbf{h}_{(\tau)} \ \longrightarrow \ \mathbf{g}_{(\mu)} \ \sqsubseteq \ \mathbf{h}_{(\tau)},$

(*d*) $\mathbf{g}_{(\mu)} \ \sqsubseteq \ \mathbf{f}_{(\sigma)} \ \ \& \ \ \mathbf{h}_{(\tau)} = \ \mathbf{k}_{(\omega)} \circ \mathbf{f}_{(\sigma)} \ \longrightarrow \ \mathbf{k}_{(\omega)} \circ \mathbf{g}_{(\mu)} \ \sqsubseteq \ \mathbf{h}_{(\tau)},$

(*e*) $\mathbf{g}_{(\mu)} \ \sqsubseteq \ \mathbf{k}_{(\omega)} \ \ \& \ \ \mathbf{h}_{(\tau)} = \ \mathbf{k}_{(\omega)} \circ \mathbf{f}_{(\sigma)} \ \longrightarrow \ \mathbf{g}_{(\mu)} \circ \mathbf{f}_{(\sigma)} \ \sqsubseteq \ \mathbf{h}_{(\tau)}.$

5. Tuples, Functions & Products

Unlike the usual set-theoretic definition of function as a set of ordered pairs, a function will be defined as a morphism with a restricted behavior, one compatible with the accepted behavior of classically defined functions.

Definition 5.1. Function: $\mathbf{f}_{(\sigma)}$ is a function $\iff$

$$(\forall y)\big(\ Sing(y) \wedge \ \mathbf{f}\big[y\big]_{\sigma} \neq \varnothing \rightarrow Sing(\mathbf{f}\big[y\big]_{\sigma}) \ \big).$$

It remains to be shown that this non-classical definition will behave as expected when used in familiar set-theoretic contexts.

Definition 5.2. Scope Projection:

$$\boldsymbol{\rho}_s(\mathbf{z}) = x \iff x \in_s \mathbf{z} \ \& \ (\forall y)(y \in_s \mathbf{z} \rightarrow y = x).$$

Definition 5.3. σ-Scopes: $\mathcal{S}_\sigma(\mathbf{x}) = \{a^\sigma\colon (\exists s)\big(\ a \in_s \mathcal{S}(\mathbf{x})\big)\}.$

Definition 5.4. Natural Numbers: $\mathbb{N} = \ \{\mathbf{1}, \ \mathbf{2}, \ \mathbf{3}...\}$, where

$$\mathbf{1} = \{\varnothing^{\varnothing}\}, \quad \mathbf{2} = !\mathbf{1} = !\{\varnothing^{\varnothing}\} = \Big\{ \ \varnothing^{\varnothing}, \ \{\varnothing^{\varnothing}\}^{\varnothing} \ \Big\}, \quad \mathbf{3} = !\mathbf{2}, ...$$

Definition 5.5. n-tuple: $< x_1, x_2, ..., x_n > \ = \ \{ \ x_1^1, x_2^2, ..., x_n^n \ \}.$

Consequence 5.1. $\boldsymbol{\rho}_i(< x_1, x_2, .., x_n >) = x_i,\ for\ \ 1 \le i \le n.$

Definition 5.6. Tuple Degree: $tup(x) = n \iff \mathbf{Sf}(x)\ \&\ \mathcal{S}_{\varnothing}(x) = \mathbb{N}(n)$.

Definition 5.7. Concatenation: $x \cdot y =$

$$\Big\{z^i\colon (\exists n, m)\Big(tup(x) = n\ \&\ tup(y) = m\ \&\ \big(1 \le i \le n \rightarrow z = \boldsymbol{\rho}_i(x)\big)$$
$$\&\ \big(1 \le i - n \le m \rightarrow z = \boldsymbol{\rho}_{i-n}(y)\big)\Big)\Big\}.$$

EXAMPLE: $< x_1, .., x_n > \cdot < y_1, .., y_m > \ = < x_1, .., x_n, y_1, .., y_m >$.

Note: $tup(x) = n\ \ \&\ \ tup(y) = m \longrightarrow tup(x \cdot y) = n + m.$

Definition 5.8. Concatenation Product: $\mathbf{A} \dot{\times} \mathbf{B} =$

$\big\{(x \cdot y)^{(s \cdot t)}\colon x \in_s \mathbf{A}\ \&\ y \in_t \mathbf{B}\ \&\ tup(x) = tup(s) \neq \varnothing\ \&\ tup(y) = tup(t) \neq \varnothing\ \big\}.$

Notice that since both the elements and scopes in the above definition are defined by concatenation, they both must be tuples. Notice also unlike the Cartesian product in CST, $\mathbf{A} \dot{\times} \mathbf{B} \dot{\times} \mathbf{C}\ =\ \mathbf{A} \dot{\times} (\mathbf{B} \dot{\times} \mathbf{C})\ =\ (\mathbf{A} \dot{\times} \mathbf{B}) \dot{\times} \mathbf{C}$.

Definition 5.9. Bind: $\mathbf{A}^{(a)} = \{\{x^a\}^{\{s^a\}}\colon x \in_s \mathbf{A}\}$.

A *Map set* is a set of elements with scopes such that both share the same scope sets and both the element sets and their scope sets have no duplicate scope values.

Definition 5.10. Map Set:

$$\mathbf{Mf}(\mathbf{Q}) \iff (\forall \mathbf{z}, w)(\mathbf{z} \in_w \mathbf{Q} \rightarrow [\mathcal{S}(\mathbf{z}) = \mathcal{S}(w)\ \&\ \mathbf{Sf}(\mathbf{z})\ \&\ \mathbf{Sf}(w)]).$$

Theorem 5.11. *For all* $\mathbf{A}$ *and* a, $\mathbf{A}^{(a)} \neq \varnothing \rightarrow \mathbf{Mf}(\mathbf{A}^{(a)})$.

Definition 5.12. Map Scopes:

$$(\forall \mathbf{f})(\mathbf{Mf}(\mathbf{f})) \rightarrow \mathbf{Ms}(\mathbf{f}) = \{s^{\varnothing} : (\exists x, t)(x \in_t \mathbf{f}\ \&\ \mathcal{S}(x) = \mathcal{S}(t)\ \&\ s \in_s \mathcal{S}(x))\}.$$

Definition 5.13. Re-Scope:

$$\mathbf{z}^{r(a)} = \{x^s : (\exists t)(x \in_t \mathbf{z}\ \&\ \boldsymbol{\rho}_t(a) = s\}$$

Definition 5.14. Re-Map:

$$\mathbf{Q}^{[a]} = \{x^s : (\exists y, t)(y \in_t \mathbf{Q}\ \&\ x = y^{r(a)}\ \&\ s = t^{r(a)}\}$$

Definition 5.15. Cross Product:

$$\mathbf{A} \otimes \mathbf{B} = \{\mathbf{z}^w\colon (\exists x, s, y, t)[(x \in_s \mathbf{A}\ \&\ y \in_t \mathbf{B})\ \&\ (\mathbf{z} = x \cup y\ \&\ w = s \cup t)]\}.$$

Definition 5.16. Cartesian Product: $\mathbf{A} \times \mathbf{B} = \mathbf{A}^{(1)} \otimes \mathbf{B}^{(2)}$.

Definition 5.17. Indexed Set Product: $\prod\limits_{i \in\ I} \mathbf{A}_i\ = \mathbf{A}_1^{(1)} \otimes \mathbf{A}_2^{(2)} \otimes ... \otimes \mathbf{A}_n^{(n)} \otimes ...$

Consequence 5.2. $A \times B \times C = A^{(1)} \otimes B^{(2)} \otimes C^{(3)}.$

Consequence 5.3. $A \times (B \times C) = A^{(1)} \otimes (B^{(1)} \otimes C^{(2)})^{(2)}.$

Consequence 5.4. $(A \times B) \times C = (A^{(1)} \otimes B^{(2)})^{(1)} \otimes C^{(2)}.$

The above reflect classical behavior for Kuratowski defined ordered pairs.

Definition 5.18. Labeled Set Product:

Given $(\forall i, j)(a_i \neq a_j)$. *then* $\prod\limits_{a_i \in\ L} \mathbf{A}_i\ = \mathbf{A}_1^{(a_1)} \otimes \mathbf{A}_2^{(a_2)} \otimes ... \otimes \mathbf{A}_n^{(a_n)} \otimes ...$

Definition 5.19. Natural Product:

$$\mathbf{z} \in_w \prod \mathbf{Q} \iff \mathcal{X}(\mathbf{z})\ \&\ (\mathcal{S}(\mathbf{z}) = \mathcal{S}(\mathbf{w}))\ \&\ \mathbf{Sf}(\mathbf{z}, \mathbf{w}, \mathbf{Q})\ \&$$
$$(\forall x, s, y)(x \in_s \mathbf{z}\ \ \&\ \ y \in_s w \rightarrow (\exists A)(A \in_s \mathbf{Q}\ \ \&\ \ x \in_y A)).$$

Theorem 5.20. $\prod \mathbf{Q} \neq \varnothing \ \rightarrow \ (\forall A, a)(A \in_a \mathbf{Q} \rightarrow \ \prod \mathbf{Q} \otimes A^{(a)} = \prod \mathbf{Q})$.

Definition 5.21. Cartesian Union:

$$\mathbf{A} \uplus \mathbf{B} = \{\mathbf{z}^s \colon (\exists x, a, y, b)(x \in_a \mathbf{A} \ \& \ y \in_b \mathbf{B} \ \& \ \mathbf{z} = x \cup y \ \& \ s \in \{a, b\})\}.$$

Definition 5.22. Discrete Union: $\mathbf{A} \oplus \mathbf{B} = (\mathbf{A} \sim \mathbf{B}) \cup (\mathbf{B} \sim \mathbf{A})$.

Definition 5.23. Disjoint Sum: $\mathbf{A} + \mathbf{B} = \mathbf{A}^{(1)} \oplus \mathbf{B}^{(2)}$.

Definition 5.24. Labeled Disjoint Union:
Given $(\forall i, j)(a_i \neq a_j)$. *then* $\coprod_{a_i \in \ L} \mathbf{A}_i \ = \mathbf{A}_1^{(a_1)} \oplus \mathbf{A}_2^{(a_2)} \oplus ... \oplus \mathbf{A}_n^{(a_n)} \oplus ...$

Definition 5.25. Mutually Disjoint Map Sets:
$\mathbf{Mfd}(\mathbf{A}, \mathbf{B}) \iff \mathbf{Mf}(\mathbf{A}) \ \& \ \mathbf{Mf}(\mathbf{B}) \ \& \ \mathbf{Ms}(\mathbf{A}) \cap \mathbf{Ms}(\mathbf{B}) = \varnothing$

Theorem 5.26. $(\forall i, j)(\mathbf{A}_i \in_i \prod \mathbf{A} \ \& \ \mathbf{A}_j \in_j \prod \mathbf{A} \ \& \ i \neq j \rightarrow \mathbf{Mfd}(\mathbf{A}_i, \mathbf{A}_j)$

Theorem 5.27. $(\forall i, j)(\mathbf{A}_i \in_i \coprod \mathbf{A} \ \& \ \mathbf{A}_j \in_j \coprod \mathbf{A} \ \& \ i \neq j \rightarrow \mathbf{Mfd}(\mathbf{A}_i, \mathbf{A}_j)$

6. Relations

The intent is to define a general concept of *relations* that is compatible with the spirit of classical relations, but not restricted to binary codifications.

Definition 6.1. Relation
$\mathbf{R}$ is a relation under $\mathbf{L} \iff (\forall x, s)(x \in_s \mathbf{R} \rightarrow \mathcal{S}(x) = \mathbf{L})$.

Definition 6.2. Normal Relation
$\mathbf{R}$ is a normal relation under $\mathbf{L} \iff$
$(\forall x, s)(x \in_s \mathbf{R} \rightarrow \mathcal{S}(x) = \mathbf{L} \ \& \ |x| = |\mathbf{L}|)$.

Definition 6.3. n-ary Relation
$\mathbf{R}$ is an n-ary relation under $\mathbf{L} \iff$
$(\forall x, s)(x \in_s \mathbf{R} \rightarrow \mathcal{S}(x) = \mathbf{L} \ \& \ |x| = |\mathbf{L}| = n)$.

Definition 6.4. CST Relation
$\mathbf{R}$ is a CST relation $\iff$ $\mathbf{R}$ is a 2-ary relation $\ \& \ \mathbf{L} =< 1, 2 >$.

Definition 6.5. Composite Relation
$\mathbf{R}$ is a composite relation under $\mathbf{L} \iff \mathbf{R} = \bigcup_i \mathbf{R}_i \ \& \ \mathbf{L} = \bigcup_i \mathbf{L}_i \ \&$
$(\forall i)(\mathbf{R}_i \in_i \mathbf{R} \ \& \ \mathbf{L}_i \in_i \mathbf{L} \rightarrow \mathbf{R}_i$ is a relation under $\mathbf{L}_i)$.

7. Relative Product

Relative Product has more personality than other operations in that given the same two operands the resultant set can have many forms. Though, in CST the operation is rather bland matching the range elements of the first operand with the domain elements of the second operand and producing a pair of the domain element of the first with the range element of the second.

For example, in CST: $\{< a, b >\}/\{< b, c >\} = \{< a, c >\}$. Following are some element combinations that are potentially more interesting:

1) $< a, b > \ \& < b, c > \longrightarrow < a, c >$,
2) $< a, b > \ \& < b, c > \longrightarrow < a, b, c >$,
3) $< a, b > \ \& < a, c > \longrightarrow < a, b, c >$,
4) $< a, b > \ \& < a, c > \longrightarrow < b, c >$,
5) $< a, c > \ \& < b, c > \longrightarrow < a, b, c >$,

6) $< a, c > \ \& < b, c > \longrightarrow < a, b >$,
7) $< a, b, c > \ \& < x, y, c, b > \longrightarrow < b, c, a, y, b, c, x, x >$,
8) $< a, b, c, w, v > \ \& < a, b, c, x, y, z > \longrightarrow < a, b, c, w, v, x, y, z >$.

All of the above are producible with the following definition.

Definition 7.1. Relative Product:

$$\mathbf{F}\Big/_{<\sigma_1,\sigma_2>}^{<\omega_1,\omega_2>}\mathbf{G} = \Big\{\mathbf{z}^{\tau}\colon (\exists x, s, y, t)(\ x \in_s \mathbf{F}\ \&\ y \in_t \mathbf{G}\ \ \&\ \ x^{/\sigma_2/} = y^{/\omega_1/}\ \ \&\ \ s^{/\sigma_2/} = t^{/\omega_1/}\ \ \&\ \ \mathbf{z} = x^{/\sigma_1/} \cup y^{/\omega_2/}\ \ \&\ \ \tau = s^{/\sigma_1/} \cup t^{/\omega_2/})\ \}.$$

Theorem 7.2. *If* $\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)}$ *exists there exists* $\mathbf{h}$ *and* τ *such that*

$$\mathbf{h}_{(\tau)} = \mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)} \iff \mathbf{h} = \mathbf{f}\Big/_{<\sigma_1,\ \sigma_2>}^{<\omega_1,\ \omega_2>}\mathbf{g} \quad \& \quad \tau = < \sigma_1,\ \omega_2 > .$$

For $\mathbf{f}\Big/_{\sigma}^{\omega}\mathbf{g}$ the following values for σ and ω support the corresponding mappings above.

1) $\sigma = < \{1^1\},\ \{2^1\} > \ \ \&\ \ \omega = < \{1^1\},\ \{2^2\} >$,
2) $\sigma = < \{1^1\},\ \{2^1\} > \ \ \&\ \ \omega = < \{1^1\},\ \{1^2, 2^3\} >$,
3) $\sigma = < \{1^1, 2^2\},\ \{1^1\} > \ \ \&\ \ \omega = < \{1^1\},\ \{2^3\} >$,
4) $\sigma = < \{2^1\},\ \{1^1\} > \ \ \&\ \ \omega = < \{1^1\},\ \{2^2\} >$,
5) $\sigma = < \{1^1\},\ \{2^1\} > \ \ \&\ \ \omega = < \{2^1\},\ \{1^2, 2^3\} >$,
6) $\sigma = < \{1^1\},\ \{2^1\} > \ \ \&\ \ \omega = < \{2^1\},\ \{1^2\} >$,
7) $\sigma = < \{2^1, 3^2, 1^3\}, \{2^1, 3^2\} > \ \ \&\ \ \omega = < \{4^1, 3^2\}\{2^4, 4^5, 3^6, 1^7, 1^8\} >$,
8) $\sigma = < \{1^1, 2^2, 3^3, 4^4, 5^5\},\ \{1^1, 2^2, 3^3\} > \ \ \&\ \ \omega = < \{1^1, 2^2, 3^3\},\ \{4^6, 5^7, 6^8\} >$.

8. Morphism & Function Spaces

An $\mathcal{M}$-Space, morphism space, is the collection of all morphisms from one given set to another.

Definition 8.1. Morphism Space, $\mathcal{M}$-Space: (where $\sigma_i = \rho_i(\sigma)$)

$$\mathcal{M}(\mathbf{A}, \mathbf{B}) = \Big\{ < \mathbf{f} >^{<\sigma>} :\ \text{Đ}_{\sigma_1}(\mathbf{f}) \Subset \mathbf{A}\ \ \&\ \text{Đ}_{\sigma_2}(\mathbf{f}) \Subset \mathbf{B}\ )\ \&\ (\forall x)\big(\ \mathbf{f}_{(\sigma)}(x) \subseteq \mathbf{B}\ \big)\Big\}.$$

Similarly, an $\mathcal{F}$-Space or function space is the collection of all functions from one given set to another.

Definition 8.2. Function Space, $\mathcal{F}$-Space:

$$\mathcal{F}(\mathbf{A}, \mathbf{B})\ =\ \Big\{\ < \mathbf{f} >^{<\sigma>} : < \mathbf{f} > \in_{<\sigma>} \mathcal{M}(\mathbf{A}, \mathbf{B})\ \ \&\ \ (\forall y)\Big(\ Sing(y)\ \ \&\ \ \mathbf{f}_{(\sigma)}(y) \neq \varnothing\ \ \rightarrow\ \ Sing\big(\mathbf{f}_{(\sigma)}(y)\big)\ \Big)\Big\}.$$

Three specific properties of functions combine to provide eight unique subsets of any given $\mathcal{F}$-Space, some of which are traditionally more interesting than others. They are: *on*, *onto*, and *one-to-one*.

Definition 8.3. $\mathcal{F}$-Space ON $\mathbf{A}$ *from* $\mathbf{A}$ *to* $\mathbf{B}$:

$$\mathcal{F}[\mathbf{A}, \mathbf{B}) = \big\{ < \mathbf{f} >^{<\sigma>} : < \mathbf{f} > \in_{<\sigma>} \mathcal{F}(\mathbf{A}, \mathbf{B})\ \ \&\ \ \text{Đ}_{\sigma_1}(\mathbf{f}) = \mathbf{A}\big\}.$$

Definition 8.4. $\mathcal{F}$-Space ONTO $\mathbf{B}$ *from* $\mathbf{A}$ *to* $\mathbf{B}$:

$$\mathcal{F}(\mathbf{A}, \mathbf{B}] = \big\{ < \mathbf{f} >^{<\sigma>} : < \mathbf{f} > \in_{<\sigma>} \mathcal{F}(\mathbf{A}, \mathbf{B})\ \ \&\ \ \text{Đ}_{\sigma_2}(\mathbf{f}) = \mathbf{B}\big\}.$$

Definition 8.5. $\mathcal{F}$-Space 1-1 *from* $\mathbf{A}$ *to* $\mathbf{B}$:

$$\mathcal{F}^{*}(\mathbf{A}, \mathbf{B}) = \big\{ < \mathbf{f} >^{<\sigma>} : < \mathbf{f} > \in_{<\sigma>} \mathcal{F}(\mathbf{A}, \mathbf{B})\ \ \&\ \ (\forall x, y)\ \big(\ Sing(x, y)\ \ \&\ \ \mathbf{f}\big[x\big]_{\sigma} = \mathbf{f}\big[y\big]_{\sigma} \neq \varnothing\ \ \rightarrow\ \ x = y\ \big)\ \big\}.$$

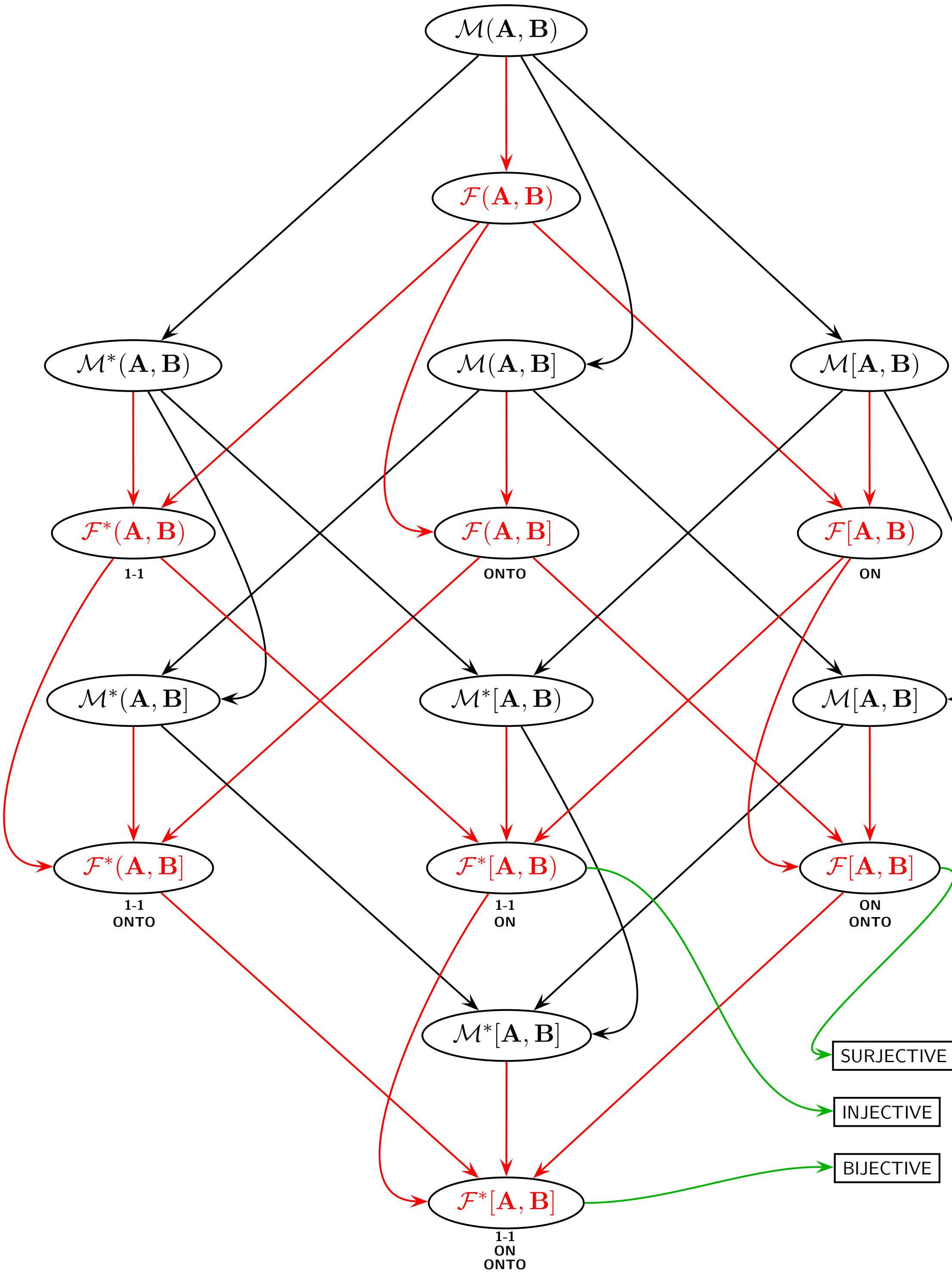


FIGURE-1: Relationship of Morphism and Function Spaces (16 of 29)

Consequence 8.1. *$\mathcal{F}$-Space Properties:*

(*a*) $\mathcal{F}[\mathbf{A},\mathbf{B}) \subseteq \mathcal{F}(\mathbf{A},\mathbf{B})$,

(*b*) $\mathcal{F}(\mathbf{A},\mathbf{B}] \subseteq \mathcal{F}(\mathbf{A},\mathbf{B})$,

(*c*) $\mathcal{F}[\mathbf{A},\mathbf{B}] \subseteq \mathcal{F}(\mathbf{A},\mathbf{B})$.

There are twenty nine different classes of spaces. Sixteen are morphisms of interest eight of which are functions. Though all have CST equivalences, only three are generally of traditional interest.

Definition 8.6. Injective $\mathcal{F}$-Space: $\mathcal{F}^*[\mathbf{A},\mathbf{B})$.

Definition 8.7. Surjective $\mathcal{F}$-Space: $\mathcal{F}[\mathbf{A},\mathbf{B}]$.

Definition 8.8. Bijective $\mathcal{F}$-Space: $\mathcal{F}^*[\mathbf{A},\mathbf{B}]$.

Assertion 8.1. $<\mathbf{A}>\in_{<\sigma_{\mathbf{A}}>} \mathcal{F}^*[\mathbf{A},\mathbf{A}]$.

Assertion 8.2. *For all* $<\mathbf{f}>\in_{<\sigma>} \mathcal{F}(\mathbf{Q},\mathbf{R})$, $\mathbf{f}_{(\sigma)} = \mathbf{f}_{(\sigma)} \circ \mathbf{I}_{\mathbf{Q}} = \mathbf{I}_{\mathbf{R}} \circ \mathbf{f}_{(\sigma)}$.

A more familiar notation for expressing a morphism (usually a function) between two sets, is the arrow, such as $\mathbf{f}_{(\sigma)}:\ \mathbf{A} \rightarrow \mathbf{B}$.

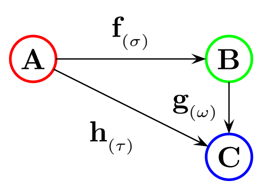
Definition 8.9. $\mathbf{f}_{(\sigma)}:\ \mathbf{A} \rightarrow \mathbf{B} \iff <\mathbf{f}>\in_{<\sigma>} \mathcal{M}(\mathbf{A},\mathbf{B})$.

Definition 8.10. $\mathbf{f}_{(\sigma)}:\ \mathbf{A} \dot{\rightarrow} \mathbf{B} \iff <\mathbf{f}>\in_{<\sigma>} \mathcal{F}^*[\mathbf{A},\mathbf{B}]$.

The above represent the two extremes from the most general morphisms to isomorphic functions. It should be noted that the above definitions impose a restriction on the allowable 'source' and 'target' sets.

9. Composition

Composition is an act of aggregating the interactive resultant behavior of multiple morphisms into a single morphism. Besides its categorical relevance for studying equivalent system behaviors, it can be used constructively to produce alternative computer programs by eliminating the intermediate operations indirectly contributing to resultant behavior. The execution of operation, **h**, in the diagram below is equivalent to executing **f** followed by an execution of **g**.



Two problems arise. It is not obvious that **h** can always be defined in terms of a given **f** and **g**, and even if it can be, it may not be of any value since morphisms are abstract mathematical objects with no authority to execute on a computer,

It needs to be shown that given any two morphisms that are compositable there is always a constructible composition and that the resultant morphism is definable in terms of structured sets.

Theorem 9.1. *For* $\mathbf{f} \mathrel{\dot{\subseteq}} \mathbf{A} \times \mathbf{B}$ *and* $\mathbf{g} \mathrel{\dot{\subseteq}} \mathbf{B} \times \mathbf{C}$ *and if* $\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)}$ *exists there exists* $\mathbf{h}$ *and* τ *such that*

$$\mathbf{h}_{(\tau)} = \mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)} \iff \mathbf{h} = \mathbf{f} \Big/^{<\sigma_1,\ \omega_2>}_{<\sigma_2,\ \omega_1>} \mathbf{g} \quad \& \quad \tau =< \sigma_1,\ \omega_2 > .$$

Proof: $\mathbf{f} \Subset \mathbf{A} \times \mathbf{B}$ and $\mathbf{g} \Subset \mathbf{B} \times \mathbf{C} \implies \sigma = \omega = \text{<< 1 >, < 2 >>}$, and $\mathbf{g}_{(\omega)} \circ \ \mathbf{f}_{(\sigma)} \implies (\forall x, s, y, t)(x \in_s \mathbf{f} \ \& \ y \in_t \mathbf{g})(x^{/\sigma_2/} = y^{/\omega_1/} \ \& \ s^{/\sigma_2/} = t^{/\omega_1/})$, thus

$$\mathbf{h}_{(\tau)} = \mathbf{g}_{(\omega)} \circ \ \mathbf{f}_{(\sigma)} \iff \mathbf{h} = \mathbf{f} \Big/^{<<1>,<2>>}_{<<2>,<1>>} \mathbf{g} \quad \& \quad \tau = \text{<< 1 >, < 2 >>}.$$

10. Kategories

A *Kategory* is a set of *objects* and *morphisms* that preserve the conditions and rules prescribed for categories. Since morphisms have been defined in such a way as to preclude their existence as members of a set, a more liberal interpretation of the term *in* needs to be considered when refereeing to a morphism being *in* a set.

Definition 10.1. A morphism, $\mathbf{f}_{(\sigma)}$, will be considered to be in a set $\mathbf{A}$ iff $(\exists x, s, v)(x \in_s \mathbf{A} \ \ \& \ \ \rho_v(x) = \mathbf{f} \ \ \& \ \ \rho_v(s) = \sigma)$.

Definition 10.2. Kategory: A set $\mathbf{Q}$ is a *Kategory* if and only if there exists $\mathbf{O}_Q$ and $\mathbf{M}_Q$ such that $\mathbf{Q} = \mathbf{O}_Q \cup \mathbf{M}_Q$ and :

(1) $\mathbf{O}_Q$ contains a collection of *OBJECTS*,

(2) $\mathbf{M}_Q$ contains a collection of (non-identity) *MORPHISMS*,

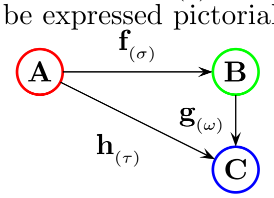
(3) For each morphism, $\mathbf{f}_{(\sigma)}$ in $\mathbf{Q}$, there exists a *DOMAIN* $\mathbf{A}$ in $\mathbf{O}_Q$, and a *CODOMAIN* $\mathbf{B}$ in $\mathbf{O}_Q$, such that $\mathbf{f}_{(\sigma)} \colon \mathbf{A} \to \mathbf{B}$.

(4) For each pair of morphisms, $\mathbf{f}_{(\sigma)}$ and $\mathbf{g}_{(\omega)}$, in $\mathbf{Q}$ such that
$\mathbf{f}_{(\sigma)} \colon \ \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{g}_{(\omega)} \colon \ \mathbf{B} \to \mathbf{C}$
there exists a *COMPOSITION*, $\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)}$, in $\mathbf{Q}$ such that
$\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)} \colon \ \mathbf{A} \to \mathbf{C}$.

(5) Each object, $\mathbf{A}$, is an *IDENTITY* morphism under $\sigma_{\mathbf{A}}$, $\mathbf{I}_{\mathbf{A}} = \mathbf{A}_{(\sigma_{\mathbf{A}})}$, such that for any morphism, $\mathbf{f}_{(\sigma)} \colon \mathbf{A} \to \mathbf{B}$, $\ \mathbf{I}_{\mathbf{B}} \circ \mathbf{f}_{(\sigma)} = \mathbf{f}_{(\sigma)}$ and $\ \mathbf{f}_{(\sigma)} \circ \mathbf{I}_{\mathbf{A}} = \mathbf{f}_{(\sigma)}$.

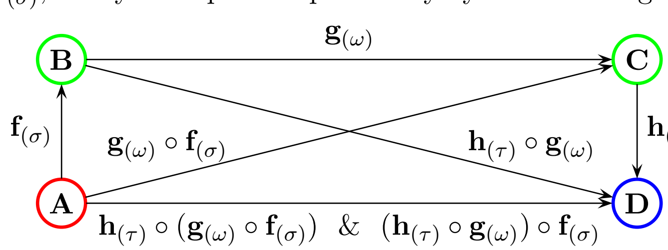
As with Categories, the formal properties relating Kategory objects and morphisms expressed above can also be expressed pictorially.

COMPOSITION: For $\mathbf{f}_{(\sigma)} \colon \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{g}_{(\omega)} \colon \mathbf{B} \to \mathbf{C}$ the composition, $\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)} \colon \mathbf{A} \to \mathbf{C}$, may be expressed pictorially by the following diagram.



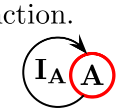
$$\mathbf{f}_{(\sigma)} : \ \mathbf{A} \to \mathbf{B}$$
$$\mathbf{g}_{(\omega)} : \ \mathbf{B} \to \mathbf{C}$$
$$\mathbf{h}_{(\tau)} : \ \mathbf{A} \to \mathbf{C}$$
$$\mathbf{h}_{(\tau)} = \ \mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)}$$

ASSOCIATIVITY: If $\mathbf{A} \xrightarrow{\mathbf{f}_{(\sigma)}} \mathbf{B} \xrightarrow{\mathbf{g}_{(\omega)}} \mathbf{C} \xrightarrow{\mathbf{h}_{(\tau)}} \mathbf{D}$, then $\mathbf{h}_{(\tau)} \circ (\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)}) = (\mathbf{h}_{(\tau)} \circ \mathbf{g}_{(\omega)}) \circ \mathbf{f}_{(\sigma)}$, may be expressed pictorially by the following diagram.

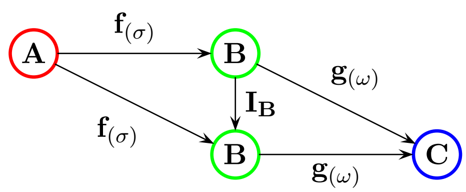


Definition 10.3. Identity Function: A map set is its own identity function.

$$\mathbf{I}_{\mathbf{A}} = \mathbf{A}_{(\sigma_{\mathbf{A}})} \iff \mathbf{Mf}(\mathbf{A}) \ \& \ \sigma_{\mathbf{A}} = \ < \cup\mathcal{S}(\mathbf{A}), \cup\mathcal{S}(\mathbf{A}) >, \ \mathbf{I}_{\mathbf{A}} \colon \mathbf{A} \to \mathbf{A}$$



IDENTITY: If $\mathbf{A} \xrightarrow{\mathbf{f}_{(\sigma)}} \mathbf{B} \xrightarrow{\mathbf{g}_{(\omega)}} \mathbf{C}$, then $\mathbf{g}_{(\omega)} \circ \mathbf{I_B} = \mathbf{g}_{(\omega)}\ and \mathbf{I_B} \circ \mathbf{f}_{(\sigma)} = \mathbf{f}_{(\sigma)}$, may be expressed pictorially by the following diagram.



These diagrams mimic the familiar category diagrams.

11. Kategories as Sets

Since a Kategory has been defined as a set and since not every set qualifies as a Kategory, it should be possible to formally distinguish those sets that qualify as Kategories form those sets that do not. $Kat(\mathbf{Q})$ will be defined to provide the distinction.

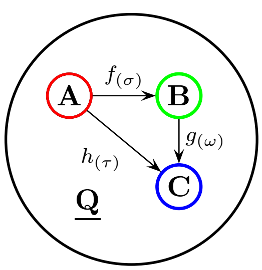
Definition 11.1. $Kat(\mathbf{Q})$: $\mathbf{Q}$ is a Kategory iff:

(a) $(\forall \mathbf{f}, \sigma)(\ <\mathbf{f}> \in_{<\sigma>} \mathbf{Q}\ \&\ (\exists \mathbf{A}, \mathbf{B})(\mathbf{A} \in_{\sigma_\mathbf{A}} \mathbf{Q}\ \&\ \mathbf{B} \in_{\sigma_\mathbf{B}} \mathbf{Q})$
$(\mathbf{Đ}_{\sigma_1}(\mathbf{f}) = \mathbf{A},$ and $\mathbf{Đ}_{\sigma_2}(\mathbf{f}) \subseteq \mathbf{B}),$

(b) $(\forall \mathbf{f}, \sigma, \mathbf{g}, \omega)\big(\ <\mathbf{f}> \in_{<\sigma>} \mathbf{Q}\ \ \&\ \ <\mathbf{g}> \in_{<\omega>} \mathbf{Q}\ \ \&\ \ (\mathbf{Đ}_{\sigma_2}(\mathbf{f}) \subseteq \mathbf{Đ}_{\omega_1}(\mathbf{g}))$
$\longrightarrow (\exists\ \mathbf{h}, \tau)(\ \ \mathbf{h}_{(\tau)} = \mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)}\ \ \&\ \ <\mathbf{h}> \in_{<\tau>} \mathbf{Q}\ )\big)$

(c) $(\forall \mathbf{f}, \sigma)\big(\ <\mathbf{f}> \in_{<\sigma>} \mathbf{Q}\ \ \&\ \ \mathbf{Đ}_{\sigma_1}(\mathbf{f}) = \mathbf{A}\ \ \&\ \ \mathbf{Đ}_{\sigma_2}(\mathbf{f}) \subseteq \mathbf{B})$
$\longrightarrow (\ \mathbf{I_B} \circ \mathbf{f}_{(\sigma)} =\ \mathbf{f}_{(\sigma)} \circ \mathbf{I_A}\ \ \&\ \ \mathbf{A} \in_{\sigma_\mathbf{A}} \mathbf{Q}\ \ \&\ \ \mathbf{B} \in_{\sigma_\mathbf{B}} \mathbf{Q}\ )\big).$

Example: $\underline{\mathbf{Q}} = Kat(\mathbf{Q})$



$$\Longrightarrow$$

$$\mathbf{Q} = \{\ <\mathbf{A}>^{<\sigma_\mathbf{A}>}, <\mathbf{B}>^{<\sigma_\mathbf{B}>}, <\mathbf{C}>^{<\sigma_\mathbf{C}>},$$
$$<f>^{<\sigma>}, <g>^{<\omega>}, <h>^{<\tau>} \}$$
$$\mathbf{Đ}_{\sigma_1}(f) = \mathbf{A} = \mathbf{Đ}_{\tau_1}(h)$$
$$\mathbf{Đ}_{\sigma_2}(f) \subseteq\ \mathbf{B} =\ \mathbf{Đ}_{\omega_1}(g)$$
$$\mathbf{Đ}_{\omega_2}(g) = \mathbf{Đ}_{\tau_2}(h) \subseteq \mathbf{C}$$
$$\mathbf{I_C} \circ h_{(\tau)} \circ \mathbf{I_A} = \mathbf{I_C} \circ g_{(\omega)} \circ \mathbf{I_B} \circ f_{(\sigma)} \circ \mathbf{I_A}$$

12. Functors & Natural Transformations

A Functor from one category to another is a function that 'preserves' domains, co-domains, identities and composites. Thus in XST, a Functor, $\mathbf{F}_{(\kappa)}$, is a function from a $Kat(\mathbf{Q})$ to a $Kat(\mathbf{R})$ that assigns:

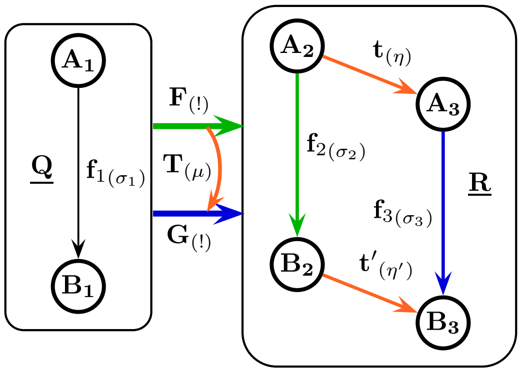
(i) to each $\mathbf{Q}$-object, $\mathbf{A}$, an $\mathbf{R}$-object, $\mathbf{F}_{(\kappa)}(\mathbf{A})$

(ii) to each $\mathbf{Q}$-morphism $f_{(\sigma)}$: $\mathbf{A} \to \mathbf{B}$ an R-morphism

$\mathbf{F}_{(\kappa)}(f_{(\sigma)})$: $\mathbf{F}_{(\kappa)}(\mathbf{A}) \to \mathbf{F}_{(\kappa)}(\mathbf{B})$, such that

(a) $\mathbf{F}_{(\kappa)}(I_{(A)}) = I_{(\mathbf{F}_{(\kappa)}(A))}$

(b) $\mathbf{F}_{(\kappa)}(g_{(\omega)} \circ f_{(\sigma)}) = \mathbf{F}_{(\kappa)}(g_{(\omega)}) \circ \mathbf{F}_{(\kappa)}(f_{(\sigma)})$, when $g_{(\omega)} \circ f_{(\sigma)}$ is defined.



$$\Longrightarrow$$

$$\mathbf{Q} = \{<\mathbf{A_1}>^{<\sigma_{\mathbf{A}}>}, <\mathbf{B_1}>^{<\sigma_{\mathbf{B}}>}, <\mathbf{f_1}>^{<\sigma_1>}\}$$
$$\mathbf{R} = \{<\mathbf{A_2}>^{<\sigma_{\mathbf{A}}>}, <\mathbf{B_2}>^{<\sigma_{\mathbf{B}}>}, <\mathbf{f_2}>^{<\sigma_2>}, <\mathbf{A_3}>^{<\sigma_{\mathbf{A}}>}, <\mathbf{B_3}>^{<\sigma_{\mathbf{B}}>}, <\mathbf{f_3}>^{<\sigma_3>}\}$$

$$Ð_{<1>}(\mathbf{F}) \;=\; Ð_{<1>}(\mathbf{G})$$
$$\mathbf{T} = \mathbf{F} \Big/^{!}_{!_*} \mathbf{G}, \quad \Phi(\mathbf{T})_{(!)}(\mathbf{A}) = \mathbf{t}_{(\eta)}, \quad \Phi(\mathbf{T})_{(!)}(\mathbf{B}) = \mathbf{t}'_{(\eta')}$$
$$<\mathbf{T}> \in_{<\nu>} \mathcal{F}[\ \mathbf{Q},\ \mathbf{R}\times\mathbf{R}\ ], \quad \nu = <<1>, <2,3>>$$
$$<\mathbf{T}> \in_{<\mu>} \mathcal{F}[\ \mathbf{F},\ \mathbf{G}\ ], \quad \mu = <<1,2>, <1,3>>$$
$$\mathbf{T}_{(\mu^*)}\big(\mathbf{F}_{(!)}(x)\big) = \mathbf{G}_{(!)}(x), \quad \mu^* = <<2>, <3>>$$
$$\Phi(\mathbf{T})_{(\nu)}(\mathbf{B})_{(\eta')} \circ \mathbf{F}_{(!)}(\mathbf{f}_{(\sigma)}) \;=\; \mathbf{G}_{(!)}(\mathbf{f}_{(\sigma)}) \circ \Phi(\mathbf{T})_{(\nu)}(\mathbf{A})_{(\eta)}$$
$$\mathbf{t}'_{(\eta')} \circ \mathbf{g}_{(\omega)} \;=\; \mathbf{g}'_{(\omega')} \circ \mathbf{t}_{(\eta)}$$

References

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[2] Blass, A.: *The Interaction Between Category Theory and Set Theory*, In Mathematical Applications of Category Theory, J, Gray, ed. A.M.S. Series in Contemporary Mathematics 30 (1984), 5-29. http://www.math.lsa.umich.edu/~ablass/interact.pdf

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Appendix A. Nested Application

Though application is well defined, sequences of applications are not. It may not be immediately apparent that even the simplest case has more that one valid interpretation. Without proper bracketing or an explicitly defined bracketing convention, its meaning is ambiguous. Therefore it must be shown that there is a case where both interpretations are non-empty and not equal to each other.

Consider the simple expression $\mathbf{f}_{(\sigma)}\mathbf{g}_{(\omega)}(x)$. There are two legitimate interpretations: $\mathbf{f}_{(\sigma)}\Big(\mathbf{g}_{(\omega)}(x)\Big)$ and $\Big(\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)})\Big)(x)$.

Example A.1. *Present a case such that:*

$$\mathbf{f}_{(\sigma)}\Big(\mathbf{g}_{(\omega)}(\mathbf{h})\Big) \neq \varnothing, \quad \Big(\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)})\Big)(\mathbf{h}) \neq \varnothing \quad \textit{and}$$
$$\mathbf{f}_{(\sigma)}\Big(\mathbf{g}_{(\omega)}(\mathbf{h})\Big) \neq \Big(\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)})\Big)(\mathbf{h}).$$

Let:
$$\mathbf{f} \;=\; \{\ <y,z>^{<,\varnothing>},\ <a,x,b,k>^{<\varnothing,\varnothing,\varnothing,\varnothing>}\ \}$$
$$\mathbf{g} \;=\; \{\ <x,y>^{<\varnothing,\varnothing>},\ <a,b>^{<\varnothing,\varnothing>}\ \},$$
$$\mathbf{p} \;=\; \{\ <x,k>^{<\varnothing,\varnothing>}\ \},$$
$$\mathbf{h} \;=\; \{\ <x>^{<\varnothing>}\ \}$$
$$\sigma \;=\; <<1,3>, <2,4>>,$$
$$\omega \;=\; <<1>, <2>>.$$

Then:
$$\mathbf{f}_{(\sigma)}\big(\{<y>^{<\varnothing>}\}\big) \;=\; \{<z>^{<\varnothing>}\}$$
$$\mathbf{f}_{(\sigma)}(\mathbf{g}) \;=\; \{<x,k>^{<\varnothing,\varnothing>}\}$$
$$\mathbf{g}_{(\omega)}(\mathbf{h}) \;=\; \{<y>^{<\varnothing>}\}$$
$$\mathbf{p}_{(\omega)}(\mathbf{h}) \;=\; \{<k>^{<\varnothing>}\} \quad \textit{and by substitution,}$$

$$\mathbf{f}_{(\sigma)}\Big(\mathbf{g}_{(\omega)}(\mathbf{h})\Big) \;=\; \mathbf{f}_{(\sigma)}\big(\{<y>^{<\varnothing>}\}\big) \;=\; \{<z>^{<\varnothing>}\} \textit{ and}$$
$$\Big(\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)})\Big)(\mathbf{h}) \;=\; \mathbf{p}_{(\omega)}(\mathbf{h}) \;=\; \{<k>^{<\varnothing>}\}.$$

Therefore, for $k \neq z$, $\mathbf{f}_{(\sigma)}\Big(\mathbf{g}_{(\omega)}(\mathbf{h})\Big) \neq \Big(\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)})\Big)(\mathbf{h})$.

Appendix B. Self Application

When functions are defined as subsets of a Cartesian product, the ability to formalize the notion of self-application, in the sense that $\mathbf{f}\big[\mathbf{f}\big] \neq \varnothing$, becomes difficult to express. Since functions under XST are not defined as subsets of a Cartesian product, the concept of self-application, in the sense of a set acting on itself, is somewhat easier to capture.

A simple, non-trivial example of self-application, difficult to express when functions are modeled as objects instead of as processes, can be generated from examining the functions from a set of cardinality 2 to itself.

Let $\mathbf{A} = \{<a>, <b>\}$, with $\mathbf{g}_1 = \{<a,a>, <b,b>\}$, $\mathbf{g}_2 = \{<a,a>, <b,a>\}$, $\mathbf{g}_3 = \{<a,b>, <b,a>\}$, and $\mathbf{g}_4 = \{<a,b>, <b,b>\}$, then for $\sigma = <<1>, <2>>$:

$$\mathbf{g}_{1(\sigma)}\Big(\{<a>\}\Big) = \{<a>\},\quad \mathbf{g}_{1(\sigma)}\Big(\{<b>\}\Big) = \{<b>\},$$
$$\mathbf{g}_{2(\sigma)}\Big(\{<a>\}\Big) = \{<a>\},\quad \mathbf{g}_{2(\sigma)}\Big(\{<b>\}\Big) = \{<a>\},$$
$$\mathbf{g}_{3(\sigma)}\Big(\{<a>\}\Big) = \{<b>\},\quad \mathbf{g}_{3(\sigma)}\Big(\{<b>\}\Big) = \{<a>\}, \text{ and}$$
$$\mathbf{g}_{4(\sigma)}\Big(\{<a>\}\Big) = \{<b>\},\quad \mathbf{g}_{4(\sigma)}\Big(\{<b>\}\Big) = \{<b>\}.$$

Let $\mathbf{f} = \{<a,a,a,b,b>, <b,b,a,a,b>\}$ and $\omega = <<1>, <1,3,4,5,2>>$, then

$$\mathbf{f}_{(\sigma)} = \mathbf{g}_{1(\sigma)},$$
$$\mathbf{f}_{(\omega)}\big(\mathbf{f}_{(\sigma)}\big) = \mathbf{g}_{2(\sigma)},$$
$$\big(\mathbf{f}_{(\omega)}\big(\mathbf{f}_{(\omega)}\big)\big)\big(\mathbf{f}_{(\sigma)}\big) = \mathbf{g}_{3(\sigma)},$$
$$\big(\big(\mathbf{f}_{(\omega)}\big(\mathbf{f}_{(\omega)}\big)\big)\big(\mathbf{f}_{(\omega)}\big)\big)\big(\mathbf{f}_{(\sigma)}\big) = \mathbf{g}_{4(\sigma)},$$

Other equalities:

$$\mathbf{f}_{(\sigma)}\big(\{<a>\}\big) = \{<a>\},\quad \mathbf{f}_{(\sigma)}\big(\{<b>\}\big) = \{<b>\},\quad \mathbf{f}_{(\sigma)} = \mathbf{I}_{\mathbf{A}}.$$

Notice that nothing in the definition of a morphism requires the resultant behavior to be functional. In fact, $\mathbf{f}_{(\tau)}$ in example [**??**] demonstrates the case where it is not.