KATEGORIES & MORFISMS (summary)

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1. Introduction

The intent is to model the properties of categories and morphisms using *Kategories* and *morfisims*.

2. Kategories

Kategories relate properties of *objects* using *morfisms*. A specific *kategory* is defined to be a collection of certain objects and morfisms following prescribed conditions and rules.

Definition 2.1. Kategory:
A collection, $\mathbf{Q}$, of *objects* and *morfisms* qualifies as a *Kategory* if and only if the following conditions and rules hold.

CONDITIONS:

(1) There exists a collection of Objects, $\boldsymbol{Obj}(\mathbf{Q})$, denoted by $\mathbf{A}, \mathbf{B}, \mathbf{C}, ..$

(2) There exists a collection of Morfisms, $\boldsymbol{Mor}(\mathbf{Q})$, denoted by $\mathbf{f}_{(\sigma)}$, $\mathbf{g}_{(\omega)}$, $\mathbf{h}_{(\tau)}, ..$

(3) For each morfism, $\mathbf{f}_{(\sigma)}$, there exists a *Domain* of $\mathbf{f}_{(\sigma)}$, $\boldsymbol{dom}(\mathbf{f}, \sigma) = \mathbf{A}$, and a *Codomain* of $\mathbf{f}_{(\sigma)}$, $\boldsymbol{cod}(\mathbf{f}, \sigma) = \mathbf{B}$, denoted by $\mathbf{f}_{(\sigma)} : \mathbf{A} \to \mathbf{B}$ *or* $\mathbf{A} \xrightarrow{\mathbf{f}_{(\sigma)}} \mathbf{B}$.

(4) For each pair of morfisms, $\mathbf{f}_{(\sigma)}$ & $\mathbf{g}_{(\omega)}$, with $\mathbf{f}_{(\sigma)} \colon \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{g}_{(\omega)} \colon \mathbf{B} \to \mathbf{C}$ there exists a *Composition*, $\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)}$, such that $\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)} : \mathbf{A} \to \mathbf{C}$.

(5) For each object, $\mathbf{A}$, there exists an *Identity*, $\mathbf{I}_{\mathbf{A}}$, such that $\mathbf{I}_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}$.

RULES:

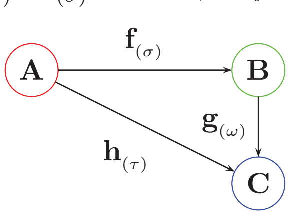
(a) ASSOCIATIVITY:
If $\mathbf{A} \xrightarrow{\mathbf{f}_{(\sigma)}} \mathbf{B} \xrightarrow{\mathbf{g}_{(\omega)}} \mathbf{C} \xrightarrow{\mathbf{h}_{(\tau)}} \mathbf{D}$, then $\mathbf{h}_{(\tau)} \circ (\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)}) = (\mathbf{h}_{(\tau)} \circ \mathbf{g}_{(\omega)}) \circ \mathbf{f}_{(\sigma)}$.

(b) IDENTITY:
If $\mathbf{A} \xrightarrow{\mathbf{f}_{(\sigma)}} \mathbf{B} \xrightarrow{\mathbf{g}_{(\omega)}} \mathbf{C}$, then $\mathbf{g}_{(\omega)} \circ \mathbf{I}_{\mathbf{B}} = \mathbf{g}_{(\omega)}$ *and* $\mathbf{I}_{\mathbf{B}} \circ \mathbf{f}_{(\sigma)} = \mathbf{f}_{(\sigma)}$.

2.1. Kat Diagrams.

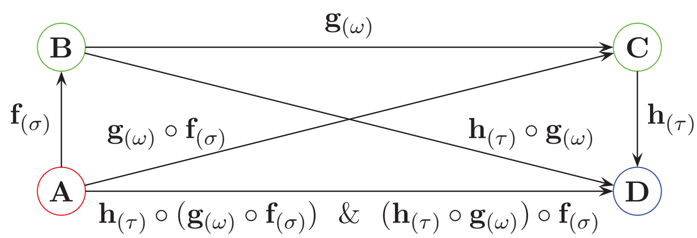
As with general Category theory, the formal properties relating Kategory objects and morfisms expressed above can also be expressed pictorially.

COMPOSITION: For $\mathbf{f}_{(\sigma)} \colon \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{g}_{(\omega)} \colon \mathbf{B} \to \mathbf{C}$ the composition, $\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)} \colon \mathbf{A} \to \mathbf{C}$, may be expressed pictorially by the following diagram.

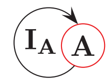


$\mathbf{f}_{(\sigma)} : \mathbf{A} \to \mathbf{B}$
$\mathbf{g}_{(\omega)} : \mathbf{B} \to \mathbf{C}$
$\mathbf{h}_{(\tau)} : \mathbf{A} \to \mathbf{C}$
$\mathbf{h}_{(\tau)} = \mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)}$

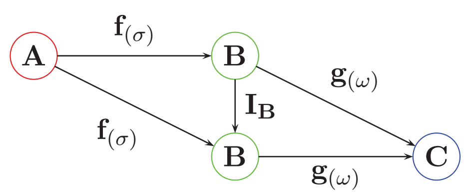
ASSOCIATIVITY: If $\mathbf{A} \xrightarrow{\mathbf{f}_{(\sigma)}} \mathbf{B} \xrightarrow{\mathbf{g}_{(\omega)}} \mathbf{C} \xrightarrow{\mathbf{h}_{(\tau)}} \mathbf{D}$, then $\mathbf{h}_{(\tau)} \circ (\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)}) = (\mathbf{h}_{(\tau)} \circ \mathbf{g}_{(\omega)}) \circ \mathbf{f}_{(\sigma)}$, may be expressed by the following diagram.



Definition 2.2. Identity: Every object, $\mathbf{A}$, has its own identity. $\mathbf{I_A}\colon \mathbf{A} \to \mathbf{A}$



IDENTITY: If $\mathbf{A} \xrightarrow{\mathbf{f}_{(\sigma)}} \mathbf{B} \xrightarrow{\mathbf{g}_{(\omega)}} \mathbf{C}$, then $\mathbf{g}_{(\omega)} \circ \mathbf{I_B} = \mathbf{g}_{(\omega)}$ *and* $\mathbf{I_B} \circ \mathbf{f}_{(\sigma)} = \mathbf{f}_{(\sigma)}$, may be expressed pictorially by the following diagram.



These diagrams mimic the familiar category diagrams.

2.2. **Kategories as Sets.**

Kategories can be modeled by extended sets.

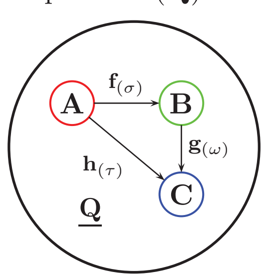
Definition 2.3. $Kat(\mathbf{Q})$: $\mathbf{Q}$ is a Kategory iff:

(a) $(\forall \mathbf{f}, \sigma)(\ \mathbf{f} \in_\sigma \boldsymbol{Mor}(\mathbf{Q})\ \&\ (\exists \mathbf{A}, \mathbf{B})(\mathbf{A} \in_{\mathbf{I_A}} \boldsymbol{Obj}(\mathbf{Q})\ \ \&\ \ \mathbf{B} \in_{\mathbf{I_B}} \boldsymbol{Obj}(\mathbf{Q}))$
$\quad (\ \boldsymbol{dom}(\mathbf{f}, \sigma) = \mathbf{A},\ \text{and}\ \ \boldsymbol{cod}(\mathbf{f}, \sigma) = \mathbf{B}\ ),$

(b) $(\forall \mathbf{f}, \sigma, \mathbf{g}, \omega)\big(\ \mathbf{f} \in_\sigma \boldsymbol{Mor}(\mathbf{Q})\ \ \&\ \mathbf{g} \in_\omega \boldsymbol{Mor}(\mathbf{Q})\ \ \&\ \ \boldsymbol{cod}(\mathbf{f}, \sigma) = \boldsymbol{dom}(\mathbf{g}, \omega)\ )$
$\quad \longrightarrow\ (\exists\ \mathbf{h}, \tau)(\ \ \mathbf{h}_{(\tau)} = \mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)}\ \ \&\ \ \mathbf{h} \in_\tau \boldsymbol{Mor}(\mathbf{Q})\ )\big)$

(c) $(\forall \mathbf{f}, \sigma)\big(\ \mathbf{f} \in_\sigma \boldsymbol{Mor}(\mathbf{Q})\ \ \&\ \ \boldsymbol{dom}(\mathbf{f}, \sigma) = \mathbf{A}\ \ \&\ \ \boldsymbol{cod}(\mathbf{f}, \sigma) = \mathbf{B})$
$\quad \longrightarrow\ (\ \mathbf{I_B} \circ \mathbf{f}_{(\sigma)} =\ \mathbf{f}_{(\sigma)} \circ \mathbf{I_A}\ \ \&\ \ \mathbf{A} \in_{\mathbf{I_A}} \boldsymbol{Obj}(\mathbf{Q})\ \ \&\ \ \mathbf{B} \in_{\mathbf{I_B}} \boldsymbol{Obj}(\mathbf{Q})\ )\big).$

Example: $Kat(\mathbf{Q})$



$\Longrightarrow$

$\mathbf{Q} = \{\ \langle \boldsymbol{Obj}(\mathbf{Q}), \boldsymbol{Mor}(\mathbf{Q}) \rangle\ \}$
$\boldsymbol{Obj}(\mathbf{Q}) = \{\mathbf{A}^{\mathbf{I_A}}, \mathbf{B}^{\mathbf{I_B}}, \mathbf{C}^{\mathbf{I_C}}\ \}\quad \boldsymbol{Mor}(\mathbf{Q}) = \{\mathbf{f}^\sigma, \mathbf{g}^\omega, \mathbf{h}^\tau\ \}$
$\boldsymbol{dom}(\mathbf{f}, \sigma) = \boldsymbol{dom}(\mathbf{h}, \tau) = \mathbf{A}$
$\boldsymbol{cod}(\mathbf{f}, \sigma) = \boldsymbol{dom}(\mathbf{g}, \omega) = \mathbf{B}$
$\boldsymbol{cod}(\mathbf{g}, \omega) = \boldsymbol{cod}(\mathbf{h}, \tau) = \mathbf{C}$
$\mathbf{h}_{(\tau)} = \mathbf{g}_{(\omega)} \circ\ \mathbf{f}_{(\sigma)},\ \tau = \langle \sigma_1, \omega_2 \rangle.$
$\mathbf{I_C} \circ \mathbf{h}_{(\tau)} \circ \mathbf{I_A} = \mathbf{I_C} \circ \mathbf{g}_{(\omega)} \circ \mathbf{I_B} \circ \mathbf{f}_{(\sigma)} \circ \mathbf{I_A}$

Notice that by this construction $\boldsymbol{Obj}(\mathbf{Q})$ is also the collection of *Identities* of $Kat(\mathbf{Q})$.