SUMMARY OF AXIOMS FOR EXTENDED SET THEORY

XST (extended set theory) differs from CST (classical set theory) by assuming a ternary membership conditional instead of a binary membership conditional. Where the membership condition of CST establishes $\mathbf{x}$ to be an element of a set $\mathbf{A}$ whenever the truth-functional $\Gamma_A(\mathbf{x})$ is true, an XST membership conditional establishes $\mathbf{x}$ to be a $\mathbf{y}$-element of a set $\mathbf{A}$ whenever the truth-functional $\Gamma_A(\mathbf{x}, \mathbf{y})$ is true. The $\mathbf{y}$ qualifier is called the *scope* component of XST set membership. Since CST sets have no scopes they can be subsumed under XST by defining CST membership using the ternary membership conditional: $\Gamma_A(\mathbf{x}, \emptyset)$. Following is a summary of work presented in [Ch92].

1 Preliminary Definitions

Extended sets are collections defined by a qualified membership condition. When the qualification is *null* the extended set membership condition is equivalent to a Classical set membership condition.

Definition 1.1 *Set: Given the null set, $\emptyset$, and the extended membership predicate, $\in$, then:*

$$Y \textit{ is a set } \longleftrightarrow (\exists x, s)(\ x \in_s Y\ ) \textit{ or } Y = \emptyset, .$$

Notationally, XST sets are written as in CST except with subscripts on *element of*, $\in_s$, in conditional statements, and with superscripts on elements in extensional descriptions, $\{\ x^a, y^b, z^c\ \}$.

Definition 1.2 *Membership Convention:* $x \in Y \ \leftrightarrow\ (\exists s)(x \in_s Y)$,

Definition 1.3 *Scope Set:* $\mathcal{S}(\mathbf{A}) \ =\ \{y^y\!: \ (\exists x)(\ x \in_y \mathbf{A}\ )\}$.

EXAMPLE: $\mathcal{S}(\{a^A, b^B, c^C\}) = \{A^A, B^B, C^C\}$.

Definition 1.4 *Element Set:* $\mathcal{E}(\mathbf{A}) = \{\ x^x\!: \ (\exists y)(\ x \in_y \mathbf{A})\}$.

EXAMPLE: $\mathcal{E}(\{a^A, b^B, c^C\}) = \{a^a, b^b, c^c\}$.

Definition 1.5 *Subsets:*

$$\begin{aligned}
\mathbf{A} \subseteq \mathbf{B} &\iff (\forall x, s)\Big(\ x \in_s \mathbf{A} \ \to\ x \in_s \mathbf{B}\Big),\\
\mathbf{A} \subset \mathbf{B} &\iff \mathbf{A} \subseteq \mathbf{B} \ \&\ \mathbf{A} \neq \mathbf{B}.\\
\mathbf{A} \Subset \mathbf{B} &\iff \emptyset \neq \mathbf{A} \subset \mathbf{B},\\
\mathbf{A} \Subseteq \mathbf{B} &\iff \mathbf{A} \subseteq \mathbf{B} \ \ \&\ \ \mathbf{B} \neq \emptyset \to \mathbf{A} \neq \emptyset.
\end{aligned}$$

Following is a summary of Extended Zermelo-Fraenkel axioms with the addition of the Axiom of Choice and the Axiom of Klasses.

1.1 X: Extensions by Scope

Following are the XST axioms governing the scope component of membership.

AXIOM 1.1 Axiom of Scope Sets:
For every set, Y, there exists a scope set, S, such that:

$$(\forall x, s)(\ x \in_s Y \to\ s \in_s S) \ \&\ (\forall s)(\exists x)(s \in_s S \to\ x \in_s Y) \ \&\ (\forall x, s)(\ x \in_s S \ \to\ s = x\ ).$$

AXIOM 1.2 Axiom of Set Inversion:
For every set, Y, there exists an inversion set, $\widehat{\mathbf{Y}}$, such that: $(\forall x, y)(\ x \in_y Y \ \longleftrightarrow\ y \in_x \widehat{\mathbf{Y}})$.

Definition 1.6 *Amalgamation:* $\bigsqcup \mathbf{A} \ =\ \Big\{\ x^y\!: \ (\exists z, s)\big(\ z \in_s \mathbf{A} \ \&\ (\ x \in_y z \ \ or\ \ y \in_x z \ \ or\ \ x \in_y s \ \ or\ \ y \in_x s\ )\big)\Big\}$,

$$\textstyle\bigsqcup_0 \mathbf{A} = \mathbf{A} \cup \widehat{\mathbf{A}}, \quad \&\ \bigsqcup_{i+1} \mathbf{A} \ =\ \bigsqcup \bigsqcup_i \mathbf{A}, \quad where\ (\ i \in \{0, 1, 2, 3, ...\}).$$

AXIOM 1.3 Axiom of Nested Scopes:
For every set A, there exists a minimal set $\mathcal{S}_*(A)$ such that:

$$\mathcal{S}_*(A) \ =\ \Big\{s^s\!: \ s \in \mathcal{S}(A) \ \ or\ \ (\exists B, C)\Big(\ B \in_C A \ \&\ \big(\ s \in_s \mathcal{S}_*(B) \ \ or\ \ s \in_s \mathcal{S}_*(C)\ \big)\Big)\Big\}.$$

1.2 ZFC: Zermelo-Fraenkel & Choice

Following are the XST specifications of Zermelo-Fraenkel axioms and the XST axiom of choice.

AXIOM 1.4 Axiom of Extensionality: $(\forall A, B)\Big((\forall x, s)\Big(\, x \in_s A \;\longleftrightarrow\; x \in_s B\Big) \;\rightarrow\; A \;=\; B\Big)$.

AXIOM 1.5 Axiom of the Empty Set: $(\forall x, s)(\, x \not\in_s \emptyset)$.

AXIOM 1.6 Axiom of Unordered Pairs: $(\forall a, b, s, v)(\exists C)(\forall x, y)\,\big(x \in_y C \leftrightarrow (y = s \text{ or } y = v) \,\&\, (x = a \text{ or } x = b)\big)$.

AXIOM 1.7 Axiom of Sum Set (Union): $(\forall A)(\exists B)(\forall x, s)\Big(\, x \in_s B \;\leftrightarrow\; (\exists X, w)\Big(\, X \in_w A \;\;\&\;\; x \in_s X\,\Big)\Big)$.

AXIOM 1.8 Axiom of Infinity: $(\exists X, G)\,\big(\,\emptyset \in X \;\;\&\;\; (\forall Y)(\exists k)\,\big(\,Y \in X \;\rightarrow\; (Y \,\cup\, \{Y^k\}) \in X \,\&\, k \in G\big)\big)$.

AXIOM 1.9 Axiom of Power Set: $(\forall A)(\exists X)(\forall B)\Big(\, B \in_{\emptyset} X \;\;\longleftrightarrow\;\; B \subseteq A\,\Big)$.

AXIOM 1.10 Axiom Schema of Replacement (element):

$$\textit{If}\;\; (\forall x, y, z, s)\Big((\, x \,\epsilon_s\, A \,\&\, \Phi(x, y, s) \,\&\, \Phi(x, z, s)) \;\rightarrow\; y = z\,\Big),$$

$$\textit{then}\;\; (\exists B)(\forall y, s)\Big(\, y \,\epsilon_s\, B \leftrightarrow\; (\exists x)(\, x \,\epsilon_s\, A \,\&\, \Phi(x, y, s)\,)\Big).$$

In the above, substitute '$\widehat{A}$' for 'A' and '$\widehat{B}$' for 'B' to get replacement for scopes. (Where $\widehat{\mathbf{A}} \;=\; \{\, y^x :\; x \in_{\mathrm{y}} \mathbf{A}\;\}$.)

AXIOM 1.11 Axiom Schema of Separation: $(\exists B)(\forall x, s)\Big(\, x \,\epsilon_s\, B \leftrightarrow\; (\, x \,\epsilon_s\, A \,\&\, \Phi(x, s)\,)\Big)$.

AXIOM 1.12 Axiom of Regularity (extended): $\mathbf{A} \neq \emptyset \;\rightarrow\; (\forall i, \nexists s)\big(\,\mathbf{A} \in_s \bigsqcup_i \mathbf{A}\big)$.

This extended foundation axiom precludes both element and scope self-membership, $\mathbf{A} \in_x \mathbf{A} \;\;\&\;\; x \in_{\mathbf{A}} \mathbf{A}$, at any nested level, but allows infinite nestings: for all i, $\mathbf{A}_{i+1} \in_i \mathbf{A}_i$.

AXIOM 1.13 Axiom of Choice: *(Where* $\mathcal{X}(x) \iff (\exists\, a, s)(a \in_s x)$.*)*

$$(\forall S)\Big[(\forall x, y)\big(\, x \neq y \,\&\, x \in S \,\&\, y \in S \;\rightarrow\; \mathcal{X}(y) \,\&\, x \,\cap\, y = \emptyset \;\;\&\, \mathcal{X}(x)\,\big) \;\longrightarrow$$
$$(\exists \Psi)(\forall z)(\exists x, v)\,(\, z \in S \;\rightarrow\; (\Psi \,\cap\, z) \;=\; \{x^v\}\,)\Big].$$

1.3 K: Klasses

In principle, $\mathcal{K}$lasses are certain sets in XST that are distinguished from others by a specific relationship between nested levels of scope elements. $\mathcal{K}$lasses are intended not only to correspond closely to the familiar role of classes but to also provide a stratification of sets that allows for *big collections* of *big collections* without fear of bumping into antinomies.

Definition 1.7 *α-$\mathcal{K}$lass:* $\mathcal{K}_\alpha(A) \iff \mathcal{X}(A) \,\&\, \mathcal{S}_*\,(\cup A) \subseteq \mathcal{E}(\alpha)$.

AXIOM 1.14 Axiom of $\mathcal{K}$lasses:

$$(\forall \Phi, \alpha, W, X, s)(\exists Y)\Big(\,\mathcal{K}_\alpha(Y) \;\;\&\;\; X \in_s Y \;\;\leftrightarrow\;\; \Phi(X) \;\;\&\;\; \mathcal{S}_*(X) \subseteq \mathcal{E}(\alpha) \;\;\&\;\; s \in \big(\mathcal{E}(W) \sim \mathcal{E}(\alpha)\big)\Big).$$

The addition of a $\mathcal{K}$lass axiom is intended to over come some of the *set size* restrictions imposed by the usual ZFC set theory. In particular, functions between $\mathcal{K}$lasses can now be studied along with functions between $\mathcal{K}$lasses of $\mathcal{K}$lasses.

References

[Ch92] Childs, D L: *Axiomatic Extended Set Theory*, supported in part by Hewlett-Packard, Unpublished, 1992