

AXIOMS FOR AN EXTENDED SET THEORY

A Formal Foundation for Unified Modeling of Mathematical Objects

Abstract

This brief paper introduces extensions to ZFC axioms intended to accommodate the representation, manipulation, and behavior of a larger body of mathematical objects than are currently allowed by ZFC axioms alone. ZFC extensions support: a Skolem-suitable definition for n-tuples, infinitely nested sets, an escalating hierarchy of arbitrarily large sets, constructive definitions for categories and functors, and functions defined as the *behavior* of interacting sets.

1 INTRODUCTION

The basic idea is to extend the usual set-membership condition to have a structural component, a *scope* value. This then allows a Skolem-suitable definition for n-tuples (that behave nicely under set operations). This, in turn allows for a definition of *tuple sets* (having tuple elements with tuple scopes). Tuple sets now allow functions to be defined as the *behavior* of sets instead of as static sets, which allows sets to interact with themselves allowing $f[f] \neq \emptyset$ to support self-referencing applications. The addition of the *Klass* axiom allows arbitrarily large collections (of well-defined mathematical objects) to be well behaved under set operations. Finally, *Kategorien* (categories) can be defined and manipulated as sets.

2 XST AXIOMS

Extended set theory, XST, differs from Classical set theory, CST, by assuming a ternary membership condition instead of a binary membership condition. Where the membership condition of CST establishes x to be an element of a set A whenever the truth-functional $\Gamma_A(x)$ is true, an XST membership condition establishes x to be a y -element of a set A whenever the truth-functional $\Gamma_A(x, y)$ is true. The y qualifier is called the *scope* component of XST set membership. XST axioms and notation are fully presented in the following.

The intent of axiomatizing extensions to set theory is to provide a comprehensive formal environment for recognizing mathematical objects and their related properties. Formally, mathematical objects will be referred to as *items*. Some items will be recognized as *sets* the rest recognized as *atoms*. Sets will be distinguished by their unique and discernible properties, and atoms also will be distinguished by their unique and discernible properties.

2.1 Extended Set Preliminaries

Items that are to be recognized as sets will be those items that have *members* or the unique item, \emptyset , referred to as the *null set*. All non-set items will be considered to be atoms. For any item, x , $\mathcal{X}(x)$ will be defined to be true whenever x has members, and defined to be false otherwise. The notation $x \in_s Y$ will be used to express the membership assertion that x is an s -property element of the set Y . The value s is referred to as the *scope* of element x in set Y .

Definition 2.1 *Set:* Given the item \emptyset , the null set, and the extended membership predicate, \in , then:
an item Y is a set $\iff (\exists x, s)(x \in_s Y) \text{ or } Y = \emptyset$,

Notationally, sets can be described as in Classical set theory except with subscripts on the membership predicate, \in_s , in conditional statements, and with superscripts on elements in extensional descriptions, $\{x^a, y^b, z^c\}$.

Definition 2.2 *Union, Intersection, Relative Complement, Subsets, & Unary Union:*

$$\begin{aligned} \mathbf{A} \cup \mathbf{B} &= \left\{ x^y: x \in_y \mathbf{A} \text{ or } x \in_y \mathbf{B} \right\}, \\ \mathbf{A} \cap \mathbf{B} &= \left\{ x^y: x \in_y \mathbf{A} \text{ and } x \in_y \mathbf{B} \right\}, \\ \mathbf{A} \sim \mathbf{B} &= \left\{ x^y: x \in_y \mathbf{A} \ \& \ x \notin_y \mathbf{B} \right\}, \\ \mathbf{A} \subseteq \mathbf{B} &\iff (\forall x, s) \left(x \in_s \mathbf{A} \rightarrow x \in_s \mathbf{B} \right), \\ \mathbf{A} \subset \mathbf{B} &\iff \mathbf{A} \subseteq \mathbf{B} \ \& \ \mathbf{A} \neq \mathbf{B}, \\ \bigcup \mathbf{A} &= \left\{ x^y: (\exists z, s, t) \left(x \in_t z \ \& \ z \in_s \mathbf{A} \ \& \ y = s \cup t \right) \right\}. \end{aligned}$$

Definition 2.3 *Item Classification:*

‘ \mathbf{x} ’ is an item having members **IFF** $\mathcal{X}(\mathbf{x})$, where $\mathcal{X}(\mathbf{x}) \iff (\exists a, s)(a \in_s \mathbf{x})$, and
‘ \mathbf{x} ’ is an item with no members **IFF** $\neg \mathcal{X}(\mathbf{x})$, where $\neg \mathcal{X}(\mathbf{x}) \iff (\forall a, s)(a \notin_s \mathbf{x})$,
with the convention that: $\mathcal{X}_\emptyset(Y) \equiv \mathcal{X}(Y)$ or $Y = \emptyset$.

Definition 2.4 *Successor:* Every item has a successor which is defined by: $(\forall x)(!x = x \cup \{x^x\})$.

Items having members and \emptyset are *sets*. All other items are *atoms*. Some atoms function as *names*. Formally, given a set of atoms, \mathcal{A} , ‘ \mathbf{x} ’ is a name for a set **IFF** $\mathcal{X}(\eta(\mathbf{x})) \& \neg \mathcal{X}(\mathbf{x})$. *Natural Names* can be expressed using the item successor definition by: $\eta(1) = \{\emptyset^\emptyset\}$, $\eta(2) = !\eta(1) = \{\emptyset^\emptyset, \{\emptyset^\emptyset\}^{\{\emptyset^\emptyset\}}\}$, $\eta(3) = !\eta(2)$.

The set of all Natural names, $\mathcal{N} \subseteq \mathcal{A}$, can be defined by:

- 1) $(\forall \mathbf{x}, \mathbf{y})(\mathbf{x} \in_y \mathcal{N} \leftrightarrow \mathbf{y} = \emptyset \& \eta(\mathbf{x}) \neq \emptyset)$, and
- 2) $(\forall \mathbf{x})(\mathbf{x} \in_\emptyset \mathcal{N} \rightarrow \eta(\mathbf{x}) = \{\emptyset^\emptyset\} \text{ or } (\exists \mathbf{y})(\mathbf{y} \in_\emptyset \mathcal{N} \& \eta(\mathbf{x}) = !\eta(\mathbf{y}))$.

Given a set of atoms, \mathcal{A} containing integer looking names, it will be convenient to define variable names, ‘ $n + 1$ ’ and ‘ $1 + n$ ’, by: $\eta(n + 1) = \eta(1 + n) = !\eta(n)$, giving, for example: $\eta(4) = \eta(3 + 1) = \eta(1 + 3) = !\eta(3)$. Thus: $1 \in_\emptyset \mathcal{N}$, $2 \in_\emptyset \mathcal{N}$, $3 \in_\emptyset \mathcal{N}$, and $(n + 1) \in_\emptyset \mathcal{N}$ for all $n \in_\emptyset \mathcal{N}$, where, $\mathcal{N} = \{1^\emptyset, 2^\emptyset, \dots, n^\emptyset, \dots\}$. Natural names ‘1’ through ‘ n ’ can be represented by $\mathcal{N}(n) = \{1^\emptyset, 2^\emptyset, \dots, n^\emptyset\}$, with $\mathcal{N}(n) = \{x^\emptyset : x \in_\emptyset \mathcal{N} \& \eta(x) \subseteq \eta(n)\}$. Define $\mathcal{N}_0 = \mathcal{N} \cup \{\emptyset^\emptyset\}$, $\mathcal{N}_0(n) = \mathcal{N}(n) \cup \{\emptyset^\emptyset\}$ and $\eta(0) = \emptyset$.

Set names can be equated and ordered by mimicking the inclusion relationship (if any) that holds between the sets being named. Two names, ‘ \mathbf{x} ’ and ‘ \mathbf{y} ’ are considered equal, $\mathbf{x} = \mathbf{y}$, when their respective sets are equal, $\eta(\mathbf{x}) = \eta(\mathbf{y})$. Name ‘ \mathbf{x} ’ is considered less than name ‘ \mathbf{y} ’, $\mathbf{x} < \mathbf{y}$, when $\eta(\mathbf{x}) \subset \eta(\mathbf{y})$. This ordering relation imposed on natural names can be used to provide a natural definition for an n-tuple, as follows: $\langle x_1, x_2, \dots, x_n \rangle = \{x_1^1, x_2^2, \dots, x_n^n\}$. This is a radical departure from the usual nested set definition of n-tuple and it will not qualify as a Classical set.

2.2 Scope Axioms

Following are the XST axioms governing the scope component of membership.

AXIOM 2.1 Axiom of Scope Sets: *For each set, A , there exists a scope set, $\mathcal{S}(A)$, such that:*

$$\mathcal{X}(A) \rightarrow \mathcal{S}(A) = \{y^y : (\exists x)(x \in_y A)\} \& \neg \mathcal{X}(A) \rightarrow \mathcal{S}(A) = \emptyset.$$

AXIOM 2.2 Axiom of Nested Scopes: *For each set A , there exists a unique total scope set $\mathcal{S}_*(A)$ such that:*

$$\mathcal{X}(A) \rightarrow \mathcal{S}_*(A) = \left\{ s^s : s \in_s \mathcal{S}(A) \text{ or } (\exists B, C) \left(B \in_C A \& (s \in_s \mathcal{S}_*(B) \text{ or } s \in_s \mathcal{S}_*(C)) \right) \right\} \& \neg \mathcal{X}(A) \rightarrow \mathcal{S}_*(A) = \emptyset.$$

2.3 Zermelo-Fraenkel & Choice Axioms

Following are the XST specifications of Zermelo-Fraenkel axioms and the XST axiom of choice.

Definition 2.5 *Lazy Notation Convention:* $\mathcal{X}(a, b, \dots, z) \leftrightarrow \mathcal{X}(a) \& \mathcal{X}(b) \& \dots \& \mathcal{X}(z)$.

AXIOM 2.3 Axiom of Extensionality: $(\forall A, B) \left((\mathcal{X}(A, B)) \rightarrow \left((\forall x, s)(x \in_s A \iff x \in_s B) \rightarrow A = B \right) \right)$.

AXIOM 2.4 Axiom of the Empty Set: $(\forall x, s)(x \notin_s \emptyset)$.

Definition 2.6 *Membership Convention:* $x \in Y \leftrightarrow (\exists s)(x \in_s Y)$,

AXIOM 2.5 Axiom of Unordered Pairs: $(\forall x, y)(\exists C, \sigma)(\forall z)(z \in_\sigma C \leftrightarrow (z = x) \text{ or } (z = y))$.

AXIOM 2.6 Axiom of Union: $(\exists B)(\forall x, s) \left(x \in_s B \leftrightarrow (\exists X, t, w) \left(x \in_t X \& X \in_w A \& s = t \cup w \right) \right)$.

AXIOM 2.7 Axiom of Infinity: $(\forall s)(\exists A) \left(\emptyset \in_s A \& (\forall B)(B \in_s A \rightarrow (B \cup \{B^s\}) \in_s A) \right)$.

AXIOM 2.8 Axiom of Power Set: $(\forall A)(\exists X, \sigma)(\forall B) \left(B \in_\sigma X \iff B \subseteq A \right) \& \neg(\{\sigma^\sigma\} \subset \mathcal{S}_*(B))$.

AXIOM 2.9 Axiom Schema of Replacement:

$$(\forall s) \left(\text{If } (\forall x, y, z) \left(x \in_s A \ \& \ \Phi(x, y) \ \& \ \Phi(x, z) \rightarrow y = z \right), \right. \\ \left. \text{then } (\exists B)(\forall y) \left(y \in_s B \leftrightarrow (\exists x) \left(x \in_s A \ \& \ \Phi(x, y) \ \& \ S_*(x) = S_*(y) \right) \right) \right).$$

AXIOM 2.10 Axiom Schema of Separation: $(\forall A, s)(\exists B)(\forall x) \left(x \in_s B \leftrightarrow (x \in_s A \ \& \ \Phi(x)) \right)$.

In the above axioms, it is assumed that B is not free in Φ .

Definition 2.7 Indexed Union: $\bigcup_i(A_i) = A_1 \cup A_2 \cup \dots \cup A_n, \ i \in \mathcal{N}_0$.

Definition 2.8 n -ary Union: $\bigcup_0(A) = A, \ \bigcup_1(A) = \bigcup_0 \bigcup_0(A), \ \bigcup_{n+1}(A) = \bigcup_{n+1} \bigcup_n(A), \ n \in \mathcal{N}_0$.

AXIOM 2.11 Axiom of Regularity:

$$A \neq \emptyset \rightarrow (\forall n, B) \left(B \subseteq \bigcup_n(A) \rightarrow (\nexists t) \left(A \in_t (\mathcal{S}_*(B) \cup B) \cup \bigcup (\mathcal{S}_*(B) \cup B) \right) \right), \ n \in \mathcal{N}_0.$$

This extended axiom differs from the usual axiom in that it allows for infinitely nested sets, but still precludes both element and scope self-membership, $\mathbf{A} \in_x \mathbf{A}$ and $x \in_{\mathbf{A}} \mathbf{A}$, at any nested level, as for example:

$$\mathbf{A} = \left\{ \left\{ \left\{ x^{\{\mathbf{A}^a\}} \right\}^b \right\}^c \right\} \quad \text{or} \quad \mathbf{A} = \left\{ \left\{ \left\{ x^{\{a^{\mathbf{A}}\}} \right\}^b \right\}^c \right\} \quad \text{or} \quad \mathbf{A} = \left\{ \left\{ \left\{ \mathbf{A}^{\{x^a\}} \right\}^b \right\}^c \right\}$$

AXIOM 2.12 Axiom of Choice:

$$(\forall A, s) \left((\mathcal{X}(A) \ \& \ \emptyset \notin A) \rightarrow \left((\forall x, y) (x \neq y \ \& \ x \in_s A \ \& \ y \in_s A \rightarrow \mathcal{X}(x, y) \ \& \ x \cap y = \emptyset) \rightarrow \right. \right. \\ \left. \left. (\exists \Psi)(\forall z)(\exists x, v) \left(z \in_s A \rightarrow (\Psi \cap z) = \{x^v\} \right) \right) \right).$$

3 KCLASS AXIOM

Klasses are sets distinguished from other sets by a specific relationship between nested levels of scope values. Klasses are intended not only to correspond closely to the familiar role of classes but to also provide a stratification of sets that allows for *big collections* of *big collections* without fear of bumping into antinomies or of awakening Russell. (The stratification of classes of sets by their scope values was inspired by a paper of Andreas Blass, [B184].)

AXIOM 3.1 Axiom of KClasses: $(\forall \tau)(\exists Y) (\mathcal{X}_{\emptyset}(Y) \ \& \ (\forall a, s) (a \in_s Y \rightarrow s = \tau) \ \& \ (\forall X) (X \in_{\tau} Y \leftrightarrow (\Phi(X) \ \& \ \mathcal{S}_*(X) \subseteq \tau))$.

The addition of a Kclass axiom (Y is not free in Φ) is intended to over come some of the *set size* restrictions imposed by the usual Classical Set Theory (CST). In particular, the set of all CST sets can now be defined. In fact, since many sets of all CST sets can be defined, it will be convenient to be able to define a canonical set of all CST sets.

Sets that can be defined under a Classical set theory, CST, will be denoted by $Cst(\mathbf{A})$ being true. This requires that all scopes of elements of \mathbf{A} , at all nested levels of \mathbf{A} are \emptyset .

Definition 3.1 Classical Sets: $Cst(\mathbf{A}) \iff \mathcal{X}(\mathbf{A}) \ \& \ \mathcal{S}_*(\mathbf{A}) = \{\emptyset^{\emptyset}\}$. Note: $Cst(\mathcal{N})$.

Definition 3.2 Set of All Classical Sets: $\mathbf{SET} = \{x^y : Cst(x) \ \& \ y = !\emptyset\}$, where $!\emptyset = \{\emptyset^{\emptyset}\}$.

There is no such thing as the extended set of all extended sets, nor the Kclass of all extended sets, nor even the Kclass of all Kclass. However, the Kclass axiom justifies construction of arbitrarily large sets, some examples follow.

Definition 3.3 Set of All SETs: $\mathbf{SSET} = \{x^y : x \in_{\sigma} \mathbf{SET} \ \& \ y = !\sigma\}$.

Let $\mathbf{SET}^{(0)} = \mathbf{SET}$, $\mathbf{SET}^{(1)} = \mathbf{SSET}$, then define $\mathbf{SET}^{(n+1)} = \{x^y : x \in_{\sigma} \mathbf{SET}^{(n)} \ \& \ y = !\sigma\}$.

Definition 3.4 Power Set: $\mathcal{P}_{\sigma}(\mathbf{Q}) = \{\mathbf{A}^{\sigma} : \mathbf{A} \subseteq \mathbf{Q} \ \& \ \neg(\{\sigma^{\sigma}\} \subseteq \mathcal{S}_*(A))\}$.

In CST the Power Set of a CST set \mathbf{Q} , $\mathcal{P}_{\sigma}(\mathbf{Q})$, or set of all subsets of \mathbf{Q} , is also a CST set. This is preserved by $\mathcal{P}_{\emptyset}(\mathbf{Q})$ for $Cst(\mathbf{Q})$, which also extends to large sets with $\mathcal{P}_{!\emptyset}(\mathbf{SET})$ and $\mathcal{P}_{!\emptyset}(\mathcal{P}_{!\emptyset}(\mathbf{SET}))$

\mathcal{K} Classes are intended to correspond closely to the familiar role of classes in CST by providing mathematical handles on collections that are too *large* to be CST sets. However, \mathcal{K} Classes are sets and thus enjoy all the operational properties applied to sets. Though there is no such concept as the \mathcal{K} Class of all \mathcal{K} Classes, there are \mathcal{K} Classes of \mathcal{K} Classes. The following definition provides one mechanism for referencing a hierarchy of arbitrarily large sets.

Definition 3.5 α - \mathcal{K} Class: $\mathcal{K}_\alpha = \{x^\alpha : \mathcal{X}(x) \ \& \ \mathcal{S}_*(x) \subset \alpha\} \cup \{\emptyset^\alpha\}$.

Theorem 3.1 $(\forall \alpha)(\mathcal{X}(\alpha) \rightarrow \mathcal{X}(\mathcal{K}_\alpha))$.

Note that for all α , $\mathcal{S}_*(\mathcal{K}_\alpha) = !\alpha$, that $\mathbf{SET} \subset \mathcal{K}_{!0}$, and that $\mathcal{P}_{!0}(\mathbf{SET}) \subset \mathcal{K}_{!0}$. For arbitrary large sets \mathbf{A}_i , the n-tuple $\langle \mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n \rangle$ may not always be well defined. It can easily be shown that when each \mathbf{A}_i is a subset of some α - \mathcal{K} Class, $\langle \mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n \rangle$ is always well defined. The following are all well-defined: $\langle \mathbf{SET}, !\mathbf{SET} \rangle$, $\langle \mathcal{K}_\sigma, \mathcal{K}_\tau, \mathcal{K}_\gamma \rangle$, $\langle \langle \mathbf{SET}, !\mathbf{SET} \rangle, \mathcal{P}_{!0}(\mathcal{P}_{!0}(\mathbf{SET})) \rangle$, as are many other bizarre constructions.

Assertion 3.1 $\bigcup \mathbf{SET} = \mathbf{SET}$.

Assertion 3.2 $(\forall n)(\mathcal{P}_{!0}^{(n)}(\mathbf{SET}) \subset \mathcal{K}_{!0})$.

Assertion 3.3 $(\forall \tau)(\mathcal{K}_\tau \in_{!_\tau} \mathcal{K}_{!_\tau})$.

Assertion 3.4 $(\forall \mathbf{Q}, \tau, n)(\mathbf{Q} \in_\tau \mathcal{K}_\tau \rightarrow \mathcal{P}_\tau^{(n)}(\mathbf{Q}) \in_\tau \mathcal{K}_\tau)$.

4 TUPLES & TUPLESETS

The initial motivation for exploring extensions to ZFC set theory was the need for having a non-ambiguous membership definition for tuples. Though mathematical needs seemed well attended by variations of nested set formulations, the Kuratowski definition being the most widely used, they all manifested peculiar behavior when treated as legitimate sets. Modeling computer representations of data using tuples seemed quite natural, but proved only to be a surface convenience since formal manipulation produces curious results: $\langle a, b \rangle \cap \langle a, c \rangle = \langle a, a \rangle$, (see [Ch10]).

Definition 4.1 *Natural Numbers*: $\mathbf{N} = \{\eta(x)^\emptyset : x \in \mathcal{N}\}$.

Definition 4.2 *Tuple Notation*: $\langle x_1, x_2, \dots, x_n \rangle \equiv \{x_1^1, x_2^2, \dots, x_n^n\}$, $1, 2, 3, \dots, n \in \mathcal{N}$.

5 FUNCTIONS as BEHAVIOR

The term *function* seems to connote a sense of action or process or behavior of something applied to something. Within the framework of extended set theory, XST, the concept of a *function* is defined as a behavior of sets in terms of how specific sets react subject to their interaction with other sets. In particular, $f_{(\sigma)} : \mathbf{A} \rightarrow \mathbf{B}$ will assert that the set ‘f’ behaves as a function under set ‘ σ ’ in relating an individual member of a function domain, set ‘A’, to exactly one member of a function codomain, set ‘B’. It can be shown that all Classical set theory, CST, graph based function behavior can be expressed in terms of XST function non-graph based behavior; that the behavior of functions applied to themselves is supported; and that the concepts of Category theory can be subsumed under XST, (see [Ch10]). One interesting consequence of defining functions as a behavior is that the mathematical properties of functions need no longer be coupled with properties of a Cartesian product.

6 CONCLUSION

These axioms have been under development for some many years and could not have been completed without the contribution of Andreas Blass, who has recently completed a proof of the axiom’s consistency [B110]. It is hoped that future developments in mathematics and computer science can benefit from the foundation provided by these axioms.

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