

FUNCTIONS DEFINED BY SET BEHAVIOR

A Formal Foundation Based On Extended Set Axioms

Abstract

The term *function* seems to connote a sense of action or process or behavior of something applied to something. Within the framework of extended set theory, XST, the concept of a *function* will be defined as a behavior of sets in terms of how specific sets react subject to their interaction with other sets. In particular, $f_{(\sigma)} : \mathbf{A} \rightarrow \mathbf{B}$ will assert that the set 'f' behaves as a function under set ' σ ' in relating an individual member of the function domain, set ' \mathbf{A} ', to exactly one member of the function codomain, set ' \mathbf{B} '. It will be shown that all Classical set theory, CST, graph based function behavior can be expressed in terms of XST function non-graph based behavior; that the behavior of functions applied to themselves is supported; and that the concepts of Category theory can be subsumed under XST. A notable consequence of this approach is that the mathematical properties of functions need no longer be dependent on the mathematical properties of a Cartesian product.

1 INTRODUCTION

This paper does not purport to present anything mathematically new. It does, however, attempt to provide a more comprehensive environment for expressing and relating what is already mathematically familiar regarding functions.

Formally, mathematical objects will be referred to as *items*. Some items will be recognized as *sets*, the rest recognized as *individuals*. Sets will be distinguished by their unique and discernible properties, and individuals also will be distinguished by their unique and discernible properties. Items that are to be recognized as sets will be those items that can be determined to have *members* or the unique item, \emptyset , referred to as the *null set*. All non-set items will be considered to be individuals.

The only deviation from traditional definitions is in not defining functions as specific sets, but rather as the *behavior* of specific sets. The extended definition of function in terms of *unique set behavior* instead of being defined in terms of *unique set membership* avoids many problems emanating from graph based definitions. Specifically, the classical concern with the *self-application* of functions is avoided by focusing on the behavior of sets instead of on their anatomy. The elimination of a graph based dependence also allows for a richer definition of *morphism* allowing a strictly set-theoretic model of Categories.

For ease of presentation, axioms, definitions and related properties not directly required to support main arguments will be relegated to the appendices, and *assertions* will be used as substitutes for theorems that are proven elsewhere.

2 FUNCTIONS

The concept of *function*, as a process, seems to be richer in understanding than is reflected by formal description. A function can be modeled as an association of an input with an output generated by a process that acknowledged the input by producing the output. Formal descriptions trying to capture this concept seem obligated to impose constraints on the allowable relationships associating inputs with outputs. The most popular is that of output singularity.

For example, the square root of the square root of a real number, say 16, is understood to have four legitimate values, 2, -2, 2i, and -2i. Yet considering square root as a function does not jibe very well with formal descriptions that balk at a process producing more than one result for any given input value. A possible reason for this descriptive constraint could be the requirement that all functions be expressed as sets of ordered pairs. A more relaxed constraint would allow ternary associations between an input and two outputs.

Most likely this issue of more formally defining non-functional behavior is of little interest, and even if it were desirable to define n-ary associations, set theory does not provide a *suitable* definition for n-tuple that would allow non-binary definitions. For example, in set theories that adopt the Kuratowski codification of ordered pair the following set $\{ \langle a, b, c \rangle, \langle a, \langle b, c \rangle \rangle, \langle \langle a, b \rangle, c \rangle, \{ \langle a, 1 \rangle, \langle b, 2 \rangle, \langle c, 3 \rangle \} \}$ has cardinality 3 instead of the required 4.

The difference between the classical CST function and the extended XST function is that a CST function is a static set structure while an XST function is an active set behavior. The CST function is a subset of a Cartesian product, and